Extended finite operator calculus - an example of algebraization of analysis

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Abstract

"A Calculus of Sequences" started in 1936 by Ward constitutes the general scheme for extensions of classical operator calculus of Rota - Mullin considered by many afterwards and after Ward. Because of the notation we shall call the Ward's calculus of sequences in its afterwards elaborated form - a ψ-calculus.

The ψ-calculus in parts appears to be almost automatic, natural extension of classical operator calculus of Rota - Mullin or equivalently - of umbral calculus of Roman and Rota.

At the same time this calculus is an example of the algebraization of the analysis - here restricted to the algebra of polynomials. Many of the results of ψ-calculus may be extended to Markowsky Q-umbral calculus where Q stands for a generalized difference operator, i.e. the one lowering the degree of any polynomial by one.

This is a review article based on the recent first author contribution [1]. The article is supplemented by the short indicatory glossaries of terms and notation used by Ward, Viskov, Markowsky, Roman on one side and the Rota-oriented notation on the other side (see [33]).

KEY WORDS: extended umbral calculus , Graves-Heisenberg-Weyl algebra
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"The modern evolution... has on the whole been marked by a trend of algebraization. "

Herman Weyl

1 Introduction

We shall call the Wards calculus of sequences [2] in its afterwards last century elaborated form - a ψ-calculus because of the notation [3]-[8]. The efficiency of the Rota oriented language and our notation used has been already exemplified by easy proving of ψ-extended counterparts of all representation independent statements of ψ-calculus [3]. Here these are ψ-labelled representations of Graves-Heisenberg-Weyl (GHW)[14],[2],[17],[16] algebra of linear operators acting on the algebra P of polynomials.

As a matter of fact ψ-calculus becomes in parts almost automatic extension of Rota - Mullin calculus[9] or equivalently - of umbral calculus of Roman and Rota [9, 10, 11]. The ψ-extension relies on the notion of ∂_ψ-shift invariance of operators with ψ-derivatives ∂_ψ staying for equivalence classes representatives of special differential operators lowering degree of polynomials by one [7, 8, 12]. Many of the results of ψ-calculus may be extended to Markowsky Q-umbral calculus [12] where Q stands for arbitrary generalized difference operator, i.e. the one lowering the degree of any polynomial by one. Q-umbral calculus [12] - as we call it - includes also those generalized difference operators, which are not series in ψ-derivative ∂_ψ whatever an admissible ψ sequence would be (for - "admissible" - see next section).

The note is at the same time the operator formulation of "A Calculus of Sequences" started in 1936 by Ward [2] with the indication of the role the ψ-representations of Graves-Heisenberg-Weyl (GHW) algebra in formulation and derivation of principal statements of the ψ-extension of finite operator calculus of Rota.

Restating what was said above we observe that all statements of standard finite operator calculus of Rota are valid also in the case of ψ-extension under the almost automatic replacement of $\{D, \hat{x}, id\}$ generators of GHW by their ψ-representation correspondents $\{\partial_\psi, \hat{x}_\psi, id\}$ - see definitions 2.1 and 2.5. Naturally any specification of admissible ψ - for example the famous one defining q-calculus - has its own characteristic properties not pertaining to the standard case of Rota calculus realisation. Nevertheless the overall picture and system of statements depending only on GHW algebra is the same modulo some automatic replacements in formulas demonstrated in the sequel. The large part of that kind of job was already done in [3, 4].

The aim of this presentation is to give a general picture of the algebra of linear operators on polynomial algebra. The picture that emerges discloses the fact that any ψ-representation of finite operator calculus or equivalently - any ψ-representation of GHW algebra makes up an example of the algebraization of the analysis with generalized differential operators [12] acting on the algebra of polynomials.

We shall delimit all our considerations to the algebra P of polynomials or sometimes to the algebra of formal series. Therefore the distinction between difference and differentiation operators disappears. All linear operators on P are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited.

If all this is extended to Markowsky Q-umbral calculus [12] then many of the results of ψ-calculus may be extended to Q-umbral calculus [12]. This is achieved under the almost automatic replacement of $\{D, \hat{x}, id\}$ generators of GHW or their ψ-representation $\{\partial_\psi, \hat{x}_\psi, id\}$ by their Q-representation correspondents $\{Q, \hat{x}_Q, id\}$ - see definition 2.5.

The article is supplemented by the short indicatory glossaries of terms and notation used by Ward [2], Viskov [7], [8], Markowsky [12], Roman [28]-[31] on one side and the Rota-oriented [9]-[11] notation on the other side [3],[4].

2 Primary definitions, notation and general observations

In the following we shall consider the algebra P of polynomials $P=\mathbf{F}[\mathrm{x}]$ over the field $\mathbf{F}$ of characteristic zero. All operators or functionals studied here are to be understood as *linear* operators on P. It shall be easy to see that they are always well defined.

Throughout the note while saying "polynomial sequence $\{p_n\}_0^\infty$" we mean deg $p_n = n$; $n \geq 0$ and we adopt also the convention that deg $p_n < 0$ iff $p_n \equiv 0$.

Consider $\Im$ - the family of functions' sequences (in conformity with Viskov [7],[8],[3] notation) such that:
$\Im = \{\psi; R \supset [a,b]\ ;\ q \in [a,b]\ ;\ \psi(q) : Z \to F\ ;\ \psi_0(q) = 1\ ;\ \psi_n(q) \neq 0;\ \psi_{-n}(q) = 0;\ n \in N\}$.
We shall call $\psi = \{\psi_n(q)\}_{n\geq 0}\ ;\ \psi_n(q) \neq 0;\ n \geq 0$ and $\psi_0(q) = 1$ an admissible sequence. Let now n_ψ denotes [3, 4]

$$n_\psi \equiv \psi_{n-1}(q)\,\psi_n^{-1}(q)\,, n \geq 0.$$

Then (note that for admissible ψ, $0_\psi = 0$)

$$n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi\,(n-1)_\psi\,(n-2)_\psi\,(n-3)_\psi \ldots 2_\psi 1_\psi; \quad 0_\psi! = 1$$

$$n_\psi^{\underline{k}} = n_\psi\,(n-1)_\psi \ldots (n-k+1)_\psi,\ \left(\begin{array}{c} n \\ k \end{array}\right)_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!} \text{ and } \exp_\psi\{y\} = \sum_{k=0}^{\infty} \frac{y^k}{k_\psi!}.$$

Definition 2.1. *Let ψ be admissible. Let ∂_ψ be the linear operator lowering degree of polynomials by one defined according to $\partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$. Then ∂_ψ is called the ψ-derivative.*

Remark 2.1.
a) For any rational function R the corresponding factorial $R\,(q^n)!$ of the sequence $R(q^n)$ is defined naturally [2, 3] as it is defined for n_ψ sequence , i.e. : $R(q^n)! = R(q^n)R(q^{n-1})...R(q^1)$ The choice $\psi_n\,(q)$=$[R\,(q^n)!]^{-1}$ and $R\,(x) = \frac{1-x}{1-q}$ results in the well known q-factorial $n_q! = n_q\,(n-1)_q!$; $1_q! = 0_q! = 1$ while the ψ-derivative ∂_ψ becomes now $(n_\psi = n_q)$ the Jackson's derivative [25, 26, 27, 3, 4] ∂_q:

$$(\partial_q \varphi)\,(x) = \frac{\varphi(x)-\varphi(qx)}{(1-q)x}.$$

b) Note also that if $\psi = \{\psi_n\,(q)\}_{n\geq 0}$ and $\varphi = \{\varphi_n\,(q)\}_{n\geq 0}$ are two admissible sequences then $[\partial_\psi\ ,\ \partial_\varphi]$= 0 iff $\psi = \varphi$. Here [,] denotes the commutator of operators.

Definition 2.2. *Let $E^y\,(\partial_\psi) \equiv exp_\psi\{y\partial_\psi\} = \sum_{k=0}^{\infty} \frac{y^k \partial_\psi^k}{k_\psi!}$. $E^y\,(\partial_\psi)$ is called the generalized translation operator.*

Note 2.1. [3, 4]
$E^a\,(\partial_\psi)\,f(x) \equiv f(x +_\psi a)$; $(x +_\psi a)^n \equiv E^a\,(\partial_\psi)\,x^n$; $E^a\,(\partial_\psi)\,f = \sum_{n\geq 0} \frac{a^n}{n_\psi!}\partial_\psi^n f$;
and in general $(x +_\psi a)^n \neq (x +_\psi a)^{n-1}(x +_\psi a)$.
Note also [2] that in general $(1 +_\psi (-1))^{2n+1} \neq 0$; $n \geq 0$ though $(1 +_\psi (-1))^{2n} = 0$; $n \geq 1$.

Note 2.2. [2]
$\exp_\psi\,(x +_\psi y) \equiv exp_\psi\{x\}\exp_\psi\{y\}$ - while in general $\exp_\psi\{x+y\} \neq \exp_\psi\{x\}\exp_\psi\{y\}$.

Possible consequent utilisation of the identity $\exp_\psi\,(x +_\psi y) \equiv \exp_\psi\{x\}\exp_\psi\{y\}$ is quite encouraging. It leads among others to "ψ-trigonometry" either ψ-elliptic or ψ-hyperbolic via introducing $\cos_\psi$, $\sin_\psi$ [2], $\cosh_\psi$, $\sinh_\psi$ or in general ψ-hyperbolic functions of m-th order $\left\{h_j^{(\psi)}\,(\alpha)\right\}_{j\in Z_m}$ defined according to [13]

$$R \ni \alpha \rightarrow\ h_j^{(\psi)}\,(\alpha) = \frac{1}{m}\sum_{k\in Z_m} \omega^{-kj}\exp_\psi\left\{\omega^k \alpha\right\}\ ;\ j \in Z_m,\ \omega = \exp\left\{i\frac{2\pi}{m}\right\}.$$

where $1 < m \in N$ and $Z_m = \{0, 1, ..., m-1\}$.

Definition 2.3. *A polynomial sequence $\{p_n\}_o^\infty$ is of ψ-binomial type if it satisfies the recurrence*

$$E^y(\partial_\psi)\, p_n(x) \equiv p_n(x +_\psi y) \equiv \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)\, p_{n-k}(y).$$

Polynomial sequences of ψ-binomial type [3, 4] are known to correspond in one-to-one manner to special generalized differential operators Q, namely to those $Q = Q(\partial_\psi)$ which are ∂_ψ-shift invariant operators [3, 4]. We shall deal in this note mostly with this special case,i.e. with ψ-umbral calculus. However before to proceed let us supply a basic information referring to this general case of Q-umbral calculus.

Definition 2.4. *Let $P = \mathbf{F}[x]$. Let Q be a linear map $Q : P \to P$ such that: $\forall_{p\in P}$ $deg(Qp) = (deg\, p)-1$ (with the convention $deg\, p = -1$ means $p = const = 0$). Q is then called a generalized difference-tial operator [12] or Gel'fond-Leontiev [8] operator.*

Right from the above definitions we infer that the following holds.

Observation 2.1. Let Q be as in Definition 2.4. Let $Qx^n = \sum_{k=1}^{n} b_{n,k}x^{n-k}$ where $b_{n,1} \neq 0$ of course. Without loose of generality take $b_{1,1} = 1$. Then $\exists\, \{q_k\}_{k\geq 2} \subset \mathbf{F}$ and there exists admissible ψ such that

$$Q = \partial_\psi + \sum_{k\geq 2} q_k \partial_\psi^k \tag{2.1}$$

if and only if

$$b_{n,k} = \binom{n}{k}_\psi b_{k,k}; \quad n \geq k \geq 1;\ b_{n,1} \neq 0;\ b_{1,1} = 1. \tag{2.2}$$

If $\{q_k\}_k \geq 2$ and an admissible ψ exist then these are unique.

Notation 2.1. *In the case (2.2) is true we shall write : $Q = Q(\partial_\psi)$ because then and only then the generalized differential operator Q is a series in powers of ∂_ψ.*

Remark 2.2. Note that operators of the (2.1) form constitute a group under superposition of formal power series (compare with the formula (S) in [14]). Of course not all generalized difference-tial operators satisfy (2.1) i.e. are series just only in corresponding ψ-derivative ∂_ψ (see Proposition 3.1). For example [15] let $Q = \frac{1}{2}D\hat{x}D - \frac{1}{3}D^3$. Then $Qx^n = \frac{1}{2}n^2x^{n-1} - \frac{1}{3}n^{\underline{3}}x^{n-3}$ so according to Observation 2.1 $n_\psi = \frac{1}{2}n^2$ and there exists no admissible ψ such that $Q = Q(\partial_\psi)$.Here $\hat{x}$ denotes the operator of multipliation by x while $n^{\underline{k}}$ is a special case of $n_\psi^{\underline{k}}$ for the choice $n_\psi = n$.

Observation 2.2. ¿From theorem 3.1 in [12] we infer that generalized differential operators give rise to subalgebras $\sum_Q$ of linear maps (plus zero map of course) commuting with a given generalized difference-tial operator Q. The intersection of two different algebras $\sum_{Q_1}$ and $\sum_{Q_2}$ is just zero map added.

The importance of the above Observation 2.2 as well as the definition below may be further fully appreciated in the context of the Theorem 2.1 and the Proposition 3.1 to come.

Definition 2.5. *Let $\{p_n\}_{n\geq 0}$ be the normal polynomial sequence [12] ,i.e. $p_0(x) = 1$ and $p_n(0) = 0$; $n \geq 1$. Then we call it the ψ-basic sequence of the generalized difference-tial operator Q if in addition $Q\,p_n = n_\psi p_{n-1}$. Parallely we define a linear map $\hat{x}_Q$: $P \to P$ such that $\hat{x}_Q p_n = \frac{(n+1)}{(n+1)_\psi} p_{n+1}$; $\quad n \geq 0$. We call the operator $\hat{x}_Q$ the dual to Q operator.*

When $Q = Q(\partial_\psi) = \partial_\psi$ we write for short: $\hat{x}_{Q(\partial_\psi)} \equiv \hat{x}_{\partial_\psi} \equiv \hat{x}_\psi$ (see: Definition 2.9).
Of course $[Q, \hat{x}_Q]= id$ therefore $\{Q, \hat{x}_Q, id\}$ provide us with a continuous family of generators of GHW in - as we call it - Q-representation of Graves-Heisenberg-Weyl algebra.
In the following we shall restrict to special case of generalized differential operators Q, namely to those $Q = Q(\partial_\psi)$ which are ∂_ψ-shift invariant operators [3, 4] (see: Definition 2.6).

At first let us start with appropriate ψ-Leibnitz rules for corresponding ψ-derivatives.
ψ-Leibnitz rules:

It is easy to see that the following hold for any formal series f and g:
for ∂_q: $\quad \partial_q(f\cdot g) = (\partial_q f)\cdot g + \left(\hat{Q}f\right)\cdot(\partial_q g)$, where $\left(\hat{Q}f\right)(x) = f(qx)$;
for $\partial_R = R\left(q\hat{Q}\right)\partial_0$: $\quad \partial_R(f\cdot g)(z) = R\left(q\hat{Q}\right)\{(\partial_0 f)(z)\cdot g(z) + f(0)(\partial_0 g)(z)\}$
where - note - $R\left(q\hat{Q}\right)x^{n-1} = n_R x^{n-1}$; ($n_\psi = n_R = n_{R(q)} = R(q^n)$) and finally for $\partial_\psi = \hat{n}_\psi \partial_0$:

$$\partial_\psi(f\cdot g)(z) = \hat{n}_\psi\{(\partial_o f)(z)\cdot g(z) + f(0)(\partial_0 g)(z)\}$$

where $\hat{n}_\psi x^{n-1} = n_\psi x^{n-1}$; $n \geq 1$.

Example 2.1. Let $Q(\partial_\psi) = D\hat{x}D$, where $\hat{x}f(x) = xf(x)$ and $D = \frac{d}{dx}$. Then $\psi = \left\{\left[(n^2)!\right]^{-1}\right\}_{n\geq 0}$ and $Q = \partial_\psi$. Let $Q(\partial_\psi)\,R(q\hat{Q})\,\partial_0 \equiv \partial_R$. Then $\psi = \left\{[R(q^n)!]^{-1}\right\}_{n\geq 0}$ and $Q = \partial_\psi \equiv \partial_R$. Here $R(z)$ is any formal Laurent series;

$\hat{Q}f(x) = f(qx)$ and $n_\psi = R(q^n)$. ∂_0 is $q = 0$ Jackson derivative which as a matter of fact - being a difference operator is the differential operator of infinite order at the same time:

$$\partial_0 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n-1}}{n!} \frac{d^n}{dx^n}. \tag{2.3}$$

Naturally with the choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ the ψ-derivative ∂_ψ becomes the Jackson's derivative [25, 26, 27, 3, 4] ∂_q:

$$(\partial_q \varphi)(x) = \frac{1 - q\hat{Q}}{(1-q)} \partial_0 \varphi(x).$$

The equivalent to (2.3) form of Bernoulli-Taylor expansion one may find [16] in *Acta Eruditorum* from November 1694 under the name "*series univeralissima*".

(Taylor's expansion was presented in his "Methodus incrementorum directa et inversa" in 1715 - edited in London).

Definition 2.6. *Let us denote by $End(P)$ the algebra of all linear operators acting on the algebra P of polynomials. Let*

$$\sum\nolimits_\psi = \{T \in End(P);\ \forall\, \alpha \in \mathbf{F};\ [T, E^\alpha(\partial_\psi)] = 0\}.$$

Then $\sum_\psi$ is a commutative subalgebra of $End(P)$ of $\mathbf{F}$-linear operators. We shall call these operators T : ∂_ψ-shift invariant operators.

We are now in a position to define further basic objects of "ψ-umbral calculus" [3, 4].

Definition 2.7. *Let $Q(\partial_\psi) : P \to P$; the linear operator $Q(\partial_\psi)$ is a ∂_ψ-delta operator iff*

a) *$Q(\partial_\psi)$ is ∂_ψ - shift invariant;*

b) *$Q(\partial_\psi)(id) = const \neq 0$ where id(x)=x.*

The strictly related notion is that of the ∂_ψ-basic polynomial sequence:

Definition 2.8. *Let $Q(\partial_\psi) : P \to P$; be the ∂_ψ-delta operator. A polynomial sequence $\{p_n\}_{n\geq 0}$; deg $p_n = n$ such that:*

1) *$p_0(x) = 1$;*

2) *$p_n(0) = 0$; $n > 0$;*

3) $Q(\partial_\psi)\, p_n = n_\psi p_{n-1}$,∂_ψ*-delta operator*$Q(\partial_\psi)$*is called the* ∂_ψ*-basic polynomial sequence of the* ∂_ψ*-delta operator.*

Identification 2.1. It is easy to see that the following identification takes place: ∂_ψ-delta operator $Q(\partial_\psi) = \partial_\psi$-shift invariant generalized differential operator Q. Of course not every generalized differential operator might be considered to be such.

Note 2.3. Let $\Phi(x;\lambda) = \sum\limits_{n\geq 0} \frac{\lambda^n}{n_\psi!} p_n(x)$ denotes the ψ-exponential generating function of the ∂_ψ-basic polynomial sequence $\{p_n\}_{n\geq 0}$ of the ∂_ψ-delta operator $Q \equiv Q(\partial_\psi)$ and let $\Phi(0;\lambda) = 1$. Then $Q\Phi(x;\lambda) = \lambda\Phi(x;\lambda)$ and Φ is the unique solution of this eigenvalue problem. If in addition (2.2) is satisfied then there exists such an admissible sequence φ that $\Phi(x;\lambda) = \exp_\varphi\{\lambda x\}$ (see Example 3.1).

The notation and naming established by Definitions 2.7 and 2.8 serve the target to preserve and to broaden simplicity of Rota‘s finite operator calculus also in its extended "ψ-umbral calculus" case [3, 4]. As a matter of illustration of such notation efficiency let us quote after [3] the important Theorem 2.1 which might be proved using the fact that $\forall\ Q(\partial_\psi) \quad \exists!$ invertible $S \in \Sigma_\psi$ such that $Q(\partial_\psi) = \partial_\psi S$. (For Theorem 2.1 see also Theorem 4.3. in [12], which holds for operators, introduced by the Definition 2.5). Let us define at first what follows.

Definition 2.9. (compare with (17) in [8])
The Pincherle ψ*-derivative is the linear map* $' : \Sigma_\psi \quad\rightarrow\quad \Sigma_\psi$;

$$T\ ' = T\,\hat{x}_\psi - \hat{x}_\psi T \equiv [T,\, \hat{x}_\psi]$$

where the linear map $\hat{x}_\psi : P \rightarrow P$; *is defined in the basis* $\{x^n\}_{n\geq 0}$ *as follows*

$$\hat{x}_\psi x^n = \frac{\psi_{n+1}(q)\,(n+1)}{\psi_n(q)} x^{n+1} = \frac{(n+1)}{(n+1)_\psi} x^{n+1}; \quad n \geq 0$$

The following theorem is true [3]

Theorem 2.1. (ψ-Lagrange and ψ-Rodrigues formulas [34, 11, 12, 23, 3])
Let $\{p_n(x)\}_{n=0}^{\infty}$ *be* ∂_ψ*-basic polynomial sequence of the* ∂_ψ*-delta operator* $Q(\partial_\psi)$*. Let* $Q(\partial_\psi) = \partial_\psi S$*. Then for* $n > 0$*:*

(1) $p_n(x) = Q(\partial_\psi)'\, S^{-n-1}\, x^n$;

(2) $p_n(x) = S^{-n} x^n - \frac{n_\psi}{n}\ (S^{-n}\ )' x^{n-1}$;

(3) $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi S^{-n} x^{n-1}$;

(4) $p_n(x) = \frac{n_\psi}{n}\hat{x}_\psi(Q(\partial_\psi)\,'\,)^{-1}p_{n-1}(x)$ ($\leftarrow$ Rodrigues ψ-formula).

For the proof one uses typical properties of the Pincherle ψ-derivative [3].

Observation 2.3. [2,3]
The triples $\{\partial_\psi, \hat{x}_\psi, id\}$ for any admissible ψ-constitute the set of generators of the ψ-labelled representations of Graves-Heisenberg-Weyl (GHW) algebra [17, 18, 19]. Namely, as easily seen $[\partial_\psi, \hat{x}_\psi] = id$. (compare with Definition 2.5)

Observation 2.4. In view of the Observation 2.3 the general Leibnitz rule in ψ-representation of Graves-Heisenberg-Weyl algebra may be written (compare with 2.2.2 Proposition in [18]) as follows

$$\partial_\psi^n \quad \hat{x}_\psi^m \quad = \quad \sum_{k\geq 0}\binom{n}{k}\binom{m}{k} k!\,\hat{x}_\psi^{m-k}\,\partial_\psi^{n-k}. \tag{2.4}$$

One derives the above ψ-Leibnitz rule from ψ-Heisenberg-Weyl exponential commutation rules exactly the same way as in $\{D, \hat{x}, id\}$ GHW representation - (compare with 2.2.1 Proposition in [18]). ψ-Heisenberg-Weyl exponential commutation relations read:

$$\exp\{t\partial_\psi\}\exp\{a\hat{x}_\psi\} = \exp\{at\}\exp\{a\hat{x}_\psi\}\exp\{t\partial_\psi\}. \tag{2.5}$$

To this end let us introduce a pertinent ψ-multiplication $*_\psi$ of functions as specified below.

Notation 2.2.
$x *_\psi x^n = \hat{x}_\psi(x^n) = \frac{(n+1)}{(n+1)_\psi}x^{n+1};\quad n \geq 0$ *hence* $x *_\psi 1 = 1_\psi^{-1}x \not\equiv x$
$x^n *_\psi x = \hat{x}_\psi^n(x) = \frac{1_\psi(n+1)!}{(n+1)_\psi!}x^{n+1};\quad n \geq 0$ *hence* $1 *_\psi x = x$ *therefore*
$x *_\psi \alpha 1 = x *_\psi \alpha = \alpha 1_\psi^{-1}x$ *and* $\alpha 1 *_\psi x = \alpha *_\psi x = \alpha x$ *and*
$\forall x, \alpha \in \mathbf{F};\ f(x) *_\psi x^n = f(\hat{x}_\psi)x^n.$

For $k \neq n \quad x^n *_\psi x^k \neq x^k *_\psi x^n$ as well as $x^n *_\psi x^k \neq x^{n+k}$ - in general i.e. for arbitrary admissible ψ; compare this with $(x +_\psi a)^n \neq (x +_\psi a)^{n-1}(x +_\psi a)$.
In order to facilitate in the future formulation of observations accounted for on the basis of ψ-calculus representation of GHW algebra we shall use what follows.

Definition 2.10. *With Notation 2.2 adopted let us define the $*_\psi$ powers of x according to*
$x^{n*_\psi} \equiv x *_\psi x^{(n-1)*_\psi} = \hat{x}_\psi(x^{(n-1)*_\psi}) = x *_\psi x *_\psi \ldots *_\psi x = \frac{n!}{n_\psi!}x^n;\quad n \geq 0.$

Note that $x^{n*_\psi} *_\psi x^{k*_\psi} = \frac{n!}{n_\psi!}x^{(n+k)*_\psi} \neq x^{k*_\psi} *_\psi x^{n*_\psi} = \frac{k!}{k_\psi!}x^{(n+k)*_\psi}$ for $k \neq n$ and $x^{0*_\psi} = 1$.
This noncommutative ψ-product $*_\psi$ is deviced so as to ensure the following observations.

Observation 2.5.

(a) $\partial_\psi x^{n*_\psi} = n x^{(n-1)*_\psi}$; $n \geq 0$

(b) $\exp_\psi[\alpha x] \equiv \exp\{\alpha \hat{x}_\psi\}\mathbf{1}$

(c) $\exp[\alpha x] *_\psi (\exp_\psi\{\beta \hat{x}_\psi\}\mathbf{1}) = (\exp_\psi\{[\alpha+\beta]\hat{\mathbf{x}}_\psi\})\mathbf{1}$

(d) $\partial_\psi(x^k *_\psi \quad x^{n*_\psi}) = (Dx^k) *_\psi x^{n*_\psi} + x^k *_\psi (\partial_\psi x^{n*_\psi})$ *hence*

(e) $\partial_\psi(f *_\psi g) = (Df) *_\psi g + f *_\psi (\partial_\psi g)$; f, g - *formal series*

(f) $f(\hat{x}_\psi)g(\hat{x}_\psi)\ \mathbf{1} = f(x) *_\psi \tilde{g}(x)$; $\tilde{g}(x) = g(\hat{x}_\psi)\mathbf{1}$.

Now the consequences of Leibniz rule (e) for difference-ization of the product are easily feasible. For example the Poisson ψ-process distribution $\pi_m(x) = \frac{1}{N(\lambda,x)}\mathrm{p}_m(x)$; $\sum\limits_{m\geq 0}\mathrm{p}_m(x) = 1$ is determined by

$$\mathrm{p}_m(x) = \frac{(\lambda x)^m}{m!} *_\psi \exp_\psi[-\lambda x] \tag{2.6}$$

which is the unique solution (up to a constant factor) of the ∂_ψ-difference equations systems

$$\partial_\psi \mathrm{p}_m(x) + \lambda \mathrm{p}_m(x) = \lambda \mathrm{p}_{m-1}(x)\ m > 0\ ;\ \partial_\psi \mathrm{p}_0(x) = -\lambda \mathrm{p}_0(x) \tag{2.7}$$

Naturally $N(\lambda, x) = \exp[\lambda x] *_\psi \exp_\psi[-\lambda x]$.

As announced - the rules of ψ -product $*_\psi$ are accounted for - as a matter of fact - on the basis of ψ-calculus representation of GHW algebra. Indeed,it is enough to consult Observation 2.5 and to introduce ψ-Pincherle derivation $\hat{\partial}_\psi$ of series in powers of the symbol $\hat{x}_\psi$ as below. Then the correspondence between generic relative formulas turns out evident.

Observation 2.6. Let $\hat{\partial}_\psi \equiv \frac{\partial}{\partial \hat{x}_\psi}$ be defined according to $\hat{\partial}_\psi f(\hat{x}_\psi) = [\partial_\psi, f(\hat{x}_\psi)]$. Then $\hat{\partial}_\psi \hat{x}_\psi^n = n\hat{x}_\psi^{n-1}$; $n \geq 0$ and $\hat{\partial}_\psi \hat{x}_\psi^n\ \mathbf{1} = \partial_\psi x^{n*_\psi}$ hence $[\hat{\partial}_\psi f(\hat{x}_\psi)]\mathbf{1}\ = \partial_\psi f(x)$ where f is a formal series in powers of $\hat{x}_\psi$ or equivalently in $*_\psi$ powers of x.

As an example of application note how the solution of 2.7 is obtained from the obvious solution $\mathbf{p}_m(\hat{x}_\psi)$ of the $\hat{\partial}_\psi$-Pincherle differential equation 2.8 formulated within G-H-W algebra generated by $\{\partial_\psi, \hat{x}_\psi, id\}$

$$\hat{\partial}_\psi \mathbf{p}_m(\hat{x}_\psi) + \lambda \mathbf{p}_m(\hat{x}_\psi) = \lambda \mathbf{p}_{m-1}(\hat{x}_\psi)\, m > 0\ ;\ \partial_\psi \mathbf{p}_0(\hat{x}_\psi) = -\lambda \mathbf{p}_0(\hat{x}_\psi.) \tag{2.8}$$

Namely : due to Observation 2.5 (f) $\mathrm{p}_m(x) = \mathbf{p}_m(\hat{x}_\psi)\mathbf{1}$, where

$$\mathbf{p}_m(\hat{x}_\psi) = \frac{(\lambda \hat{x}_\psi)^m}{m!} \exp_\psi[-\lambda \hat{x}_\psi]. \tag{2.9}$$

3 The general picture

The general picture from the title above relates to the general picture of the algebra $End(P)$ of operators on P as in the following we shall consider the algebra P of polynomials $P = \mathbf{F}[\mathrm{x}]$ over the field $\mathbf{F}$ of characteristic zero.

We shall draw an over view picture of the situation distinguished by possibility to develop umbral calculus for *any* polynomial sequences $\{p_n\}_0^\infty$ instead of those of traditional binomial type only.

In 1901 it was proved [20] that every linear operator mapping P into P may be represented as infinite series in operators $\hat{x}$ and D. In 1986 the authors of [21] supplied the explicit expression for such series in most general case of polynomials in one variable (for many variables see: [22]). Thus according to Proposition 1 from [21] one has:

Proposition 3.1. Let Q be a linear operator that reduces by one the degree of each polynomial. Let $\{q_n(\hat{x})\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $\hat{T} = \sum_{n\geq 0} q_n(\hat{x}) Q^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if $\hat{T}$ is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$\hat{T} = \sum_{n\geq 0} q_n(\hat{x}) Q^n.$$

It is also a rather matter of an easy exercise to prove the Proposition 2 from [21]:

Proposition 3.2. *Let Q be a linear operator that reduces by one the degree of each polynomial. Let $\{q_n(\hat{x})\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Let a linear operator that maps polynomials into polynomials be given by*

$$\hat{T} = \sum_{n\geq 0} q_n(\hat{x}) Q^n.$$

Let $P(x;\lambda) = \sum_{n\geq 0} q_n(x)\lambda^n$ *denotes indicator of* $\hat{T}$*. Then there exists a unique formal series* $\Phi(x;\lambda)$; $\Phi(0;\lambda) = 1$ *such that:*

$$Q\Phi(x;\lambda) = \lambda\Phi(x;\lambda).$$

Then also $P(x;\lambda) = \Phi(x;\lambda)^{-1}\hat{T}\Phi(x;\lambda)$.

Example 3.1. Note that $\partial_\psi \exp_\psi\{\lambda x\} = \lambda \exp_\psi\{\lambda x\}$; $\exp_\psi[x]|_{x=0} = 1$. (*)
Hence for indicator of $\hat{T}$; $\hat{T} = \sum_{n\geq 0} q_n(\hat{x})\partial_\psi^n$ we have:

$$P(x;\lambda) = [\exp_\psi\{\lambda x\}]^{-1}\,\hat{T}\exp_\psi\{\lambda x\}. \quad (**)$$

After choosing $\psi_n(q) = [n_q!]^{-1}$ we get $\exp_\psi\{x\} = \exp_q\{x\}$. In this connection note that $\exp_0(x) = \frac{1}{1-x}$ and $\exp(x)$ are mutual limit deformations for $|x| < 1$ due to:

$$\frac{exp_0(z)-1}{z} = exp_o(z) \Rightarrow exp_0(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k;\ |z| < 1 \text{ , i.e.}$$

$$exp(x) \underset{1\leftarrow q}{\longleftarrow} exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n_q!} \underset{q\to 0}{\longrightarrow} \frac{1}{1-x}.$$

Therefore corresponding specifications of (*) such as $exp_0(\lambda x) = \frac{1}{1-\lambda x}$ or $\exp(\lambda x)$ lead to corresponding specifications of (**) for divided difference operator ∂_0 and *D operator including special cases from [21].*

To be complete let us still introduce [3, 4] an important operator $\hat{x}_{Q(\partial_\psi)}$ dual to $Q(\partial_\psi)$.

Definition 3.1. *(see Definition 2.5)*
Let $\{p_n\}_{n\geq 0}$ *be the* ∂_ψ*-basic polynomial sequence of the* ∂_ψ*-delta operator* $Q(\partial_\psi)$*. A linear map* $\hat{x}_{Q(\partial_\psi)}$*:* $P \to P$ *;* $\hat{x}_{Q(\partial_\psi)}p_n = \frac{(n+1)}{(n+1)_\psi}p_{n+1}$; $n \geq 0$ *is called the operator dual to* $Q(\partial_\psi)$*.*

Comment 3.1. Dual in the above sense corresponds to adjoint in ψ-umbral calculus language of linear functionals' umbral algebra (compare with Proposition 1.1.21 in [23]).

It is now obvious that the following holds.

Proposition 3.3. *Let* $\{q_n\left(\hat{x}_{Q(\partial_\psi)}\right)\}_{n\geq 0}$ *be an arbitrary sequence of polynomials in the operator* $\hat{x}_{Q(\partial_\psi)}$. *Then* $T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right) Q\left(\partial_\psi\right)^n$ *defines a linear operator that maps polynomials into polynomials. Conversely, if* T *is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form*

$$T = \sum_{n\geq 0} q_n\left(\hat{x}_{Q(\partial_\psi)}\right) Q\left(\partial_\psi\right)^n . \tag{3.1}$$

Comment 3.2. The pair $Q\left(\partial_\psi\right)$, $\hat{x}_{Q(\partial_\psi)}$ of dual operators is expected to play a role in the description of quantum-like processes apart from the q-case now vastly exploited [3, 4].

Naturally the Proposition 3.2 for $Q\left(\partial_\psi\right)$ and $\hat{x}_{Q(\partial_\psi)}$ dual operators is also valid.
Summing up: we have the following picture for $End(P)$ - the algebra of all linear operators acting on the algebra P of polynomials.

$$Q(P) \equiv \bigcup_{Q} \sum\nolimits_Q \subset End(P)$$

and of course $Q(P) \neq End(P)$ where the subfamily $Q(P)$ (with zero map) breaks up into sum of subalgebras $\sum_Q$ according to commutativity of these generalized difference-tial operators Q (see Definition 2.4 and Observation 2.2). Also to each subalgebra $\sum_\psi$ i.e. to each $Q\left(\partial_\psi\right)$ operator there corresponds its dual operator $\hat{x}_{Q(\partial_\psi)}$

$$\hat{x}_{Q(\partial_\psi)} \notin \sum\nolimits_\psi$$

and both $Q\left(\partial_\psi\right)$ & $\hat{x}_{Q(\partial_\psi)}$ operators are sufficient to build up the whole algebra $End(P)$ according to unique representation given by (3.1) including the ∂_ψ and $\hat{x}_\psi$ case. Summarising: for any admissible ψ we have the following general statement.
General statement:

$$End(P) = [\{\partial_\psi, \hat{x}_\psi\}] = [\{Q\left(\partial_\psi\right), \hat{x}_{Q(\partial_\psi)}\}] = [\{Q, \hat{x}_Q\}]$$

i.e. the algebra $End(P)$ is generated by any dual pair $\{Q, \hat{x}_Q\}$ including any dual pair $\{Q\left(\partial_\psi\right), \hat{x}_{Q(\partial_\psi)}\}$ or specifically by $\{\partial_\psi, \hat{x}_\psi\}$ which in turn is determined by a choice of any admissible sequence ψ.

As a matter of fact and in another words: we have bijective correspondences between different commutation classes of ∂_ψ-shift invariant operators from $End(P)$,

different abelian subalgebras $\sum_{\psi}$, distinct ψ-representations of GHW algebra, different ψ-representations of the reduced incidence algebra R(L(S)) - isomorphic to the algebra Φ_{ψ} of ψ-exponential formal power series [3] and finally - distinct ψ-umbral calculi [8, 12, 15, 24, 3, 35]. These bijective correspondences may be naturally extended to encompass also Q-umbral calculi, Q-representations of GHW algebra and abelian subalgebras $\sum_{Q}$.
(Recall: R(L(S)) is the reduced incidence algebra of L(S) where
L(S)={A; A⊂S; |A| $<\infty$}; S is countable and (L(S); $\subseteq$) is partially ordered set ordered by inclusion [11, 3]).

This is the way the Rota's devise has been carried into effect. The devise *"much is the iteration of the few"* [11] - much of the properties of literally *all* polynomial sequences - as well as GHW algebra representations - is the application of few basic principles of the ψ-umbral difference operator calculus [3, 35, 1].

$\psi-$ **Integration Remark :**
Recall : $\partial_o x^n = x^{n-1}$. ∂_o is identical with divided difference operator. ∂_o is identical with ∂_ψ for $\psi = \{\psi(q)_n\}_{n\geq 0}$; $\psi(q)_n = 1$; $n \geq 0$. Let $\hat{Q}f(x)f(qx)$.
Recall also that there corresponds to the "∂_q difference-ization" the q-integration [25, 26, 27] which is a right inverse operation to "q-difference-ization"[35, 1]. Namely

$$F(z) :\equiv \left(\int_q \varphi\right)(z) := (1-q)\, z \sum_{k=0}^{\infty} \varphi\left(q^k z\right) q^k \tag{3.2}$$

i.e.

$$F(z) \equiv \left(\int_q \varphi\right)(z) = (1-q)\, z \left(\sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi\right)(z) =$$
$$= \left((1-q)\, z \frac{1}{1-q\hat{Q}} \varphi\right)(z). \tag{3.3}$$

Of course

$$\partial_q \circ \int_q = id \tag{3.4}$$

as

$$\frac{1-q\hat{Q}}{(1-q)} \partial_0 \left((1-q)\, \hat{z} \frac{1}{1-q\hat{Q}}\right) = id. \tag{3.5}$$

Naturally (3.5) might serve to define a right inverse operation to "q-difference-ization"

$$(\partial_q \varphi)(x) = \frac{1-q\hat{Q}}{(1-q)} \partial_0 \varphi(x)$$

and consequently the "q-integration " as represented by (3.2) and (3.3). As it is well known the definite q-integral is an numerical approximation of the definite integral obtained in the $q \to 1$ limit. Following the q-case example we introduce now an R-integration (consult Remark 2.1).

$$\int_R x^n = \left(\hat{x}\frac{1}{R\left(q\hat{Q}\right)}\right) x^n = \frac{1}{R\left(q^{n+1}\right)} x^{n+1}; \quad n \geq 0 \tag{3.6}$$

Of course $\partial_R \circ \int_R = id$ as

$$R\left(q\hat{Q}\right) \partial_o \left(\hat{x}\frac{1}{R\left(q\hat{Q}\right)}\right) = id. \tag{3.7}$$

Let us then finally introduce the analogous representation for ∂_ψ difference-ization

$$\partial_\psi = \hat{n}_\psi \partial_o; \ \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \ n \geq 1. \tag{3.8}$$

Then

$$\int_\psi x^n = \left(\hat{x}\frac{1}{\hat{n}_\psi}\right) x^n = \frac{1}{(n+1)_\psi} x^{n+1}; \ n \geq 0 \tag{3.9}$$

and of course

$$\partial_\psi \circ \int_\psi = id \tag{3.10}$$

Closing Remark:

The picture that emerges discloses the fact that any ψ-representation of finite operator calculus or equivalently - any ψ-representation of GHW algebra makes up an example of the algebraization of the analysis - naturally when constrained to the algebra of polynomials. We did restricted all our considerations to the algebra P of polynomials. Therefore the distinction in-between difference and differentiation operators disappears. All linear operators on P are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited. For example $\frac{d}{dx} = \sum_{k \geq 1} \frac{d_k}{k!} \Delta^k$ where $d_k = \left[\frac{d}{dx} x^{\underline{k}}\right]_{x=0} = (-1)^{k-1}(k-1)!$ or $\Delta = \sum_{n \geq 1} \frac{\delta_n}{n!} \frac{d^n}{dx^n}$ where $\delta_n = [\Delta x^n]_{x=0} = 1$. Thus the difference and differential operators and equations are treated on the same footing. For new applications

- due to the author see [3, 4]. Our goal here was to deliver the general scheme of "ψ-umbral" algebraization of the analysis of general differential operators [12]. Most of the general features presented here are known to be pertinent to the Q representation of finite operator calculus (Viskov, Markowsky, Roman) where Q is any linear operator lowering degree of any polynomial by one . So it is most general example of the algebraization of the analysis for general differential operators [12].

4 Glossary

Here now come short indicatory glossaries of terms and notation used by Ward [2], Viskov [7, 8], Markowsky [12], Roman [28]-[32] on one side and the Rota-oriented notation on the other side. See also [33].

Ward	**Rota - oriented** (this note)
$[n]\,;\ [n]!$ basic binomial coefficient $[n,r] = \frac{[n]!}{[r]![n-r]!}$	$n_\psi;\ n_\psi!$ ψ-binomial coefficient $\binom{n}{k}_\psi \equiv \frac{n_\psi^k}{k_\psi!}$
$D = D_x$ - the operator D $D\,x^n = [n]\ x^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi\, x^n = n_\psi\, x^{n-1}$
$(x+y)^n$ $(x+y)^n \equiv \sum_{r=0}^{n} [n,r]\, x^{n-r} y^r$	$(x +_\psi y)^n$ $(x +_\psi y)^n = \sum_{k=0}^{n} \binom{n}{k}_\psi x^k y^{n-k}$

Ward	**Rota - oriented** (this note)
basic displacement symbol $E^t;\ t \in \mathbf{Z}$ $E\varphi(x) = \varphi(x+1)$ $E^t\varphi(x) = \varphi\left(x+\overline{t}\right)$	generalized shift operator $E^y\left(\partial_\psi\right) \equiv exp_\psi\{y\partial_\psi\};\, y \in \mathbf{F}$ $E(\partial_\psi)\varphi(x) = \varphi(x +_\psi 1)$ $E^y(\partial_\psi)x^n \equiv (x +_\psi y)^n$
basic difference operator $\Delta = E - id$ $\Delta = \varepsilon(D) - id = \sum\limits_{n=0}^{\infty} \frac{D^n}{[n]!} - id$	ψ-difference delta operator $\Delta_\psi = E^y(\partial_\psi) - id$

Roman	**Rota - oriented** (this note)
$t;\ tx^n = nx^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi x^n = n_\psi x^{n-1}$
$\langle t^k \vert p(x)\rangle = p^{(k)}(0)$	$[\partial_\psi^k p(x)]\vert_{x=0}$

Roman	**Rota - oriented** (this note)
evaluation functional $\epsilon_y(t) = \exp\{yt\}$ $\langle t^k \| x^n \rangle = n!\delta_{n,k}$ $\langle \epsilon_y(t) \| p(x) \rangle = p(y)$ $\epsilon_y(t)x^n = \sum_{k\geq 0} \binom{n}{k} x^k y^{n-k}$	generalized shift operator $E^y(\partial_\psi) = \exp_\psi\{y\partial_\psi\}$ $[E^y(\partial_\psi)p_n(x)]\|_{x=0} = p_n(y)$ $E^y(\partial_\psi)p_n(x) = \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)p_{n-k}(y)$
formal derivative $f'(t) \equiv \frac{d}{dt}f(t)$ $\overline{f}(t)$ compositional inverse of formal power series $f(t)$	Pincherle derivative $[Q(\partial_\psi)]` \equiv \frac{d}{d\partial_\psi}Q(\partial_\psi)$ $Q^{-1}(\partial_\psi)$ compositional inverse of formal power series $Q(\partial_\psi)$
$\theta_t;\ \ \theta_t x^n = x^{n+1};\ n \geq 0$ $\theta_t t = \hat{x}D$	$\hat{x}_\psi;\ \ \hat{x}_\psi x^n = \frac{n+1}{(n+1)_\psi}x^{n+1};\ n \geq 0$ $\hat{x}_\psi \partial_\psi = \hat{x}D = \hat{N}$
$\sum_{k\geq 0} \frac{s_k(x)}{k_\psi!}t^k =$ $[g(\overline{f}(z))]^{-1}\exp\{x\overline{f}(t)\}$ $\{s_n(x)\}_{n\geq 0}$ - Sheffer sequence for $(g(t), f(t))$	$\sum_{k\geq 0} \frac{s_k(x)}{k_\psi!}z^k =$ $s(q^{-1}(z))\exp_\psi\{xq^{-1}(z)\}$ $q(t)$, $s(t)$ indicators of $Q(\partial_\psi)$ and S_{∂_ψ}

Roman	**Rota - oriented** (this note)
$g(t)\, s_n(x) = q_n(x)$ - sequence associated for $f(t)$	$s_n(x) = S_{\partial_\psi}^{-1}\, q_n(x)$ - ∂_ψ - basic sequence of $Q(\partial_\psi)$
The expansion theorem: $h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \vert p_k(x)\rangle}{k!} f(t)^k$ $p_n(x)$ - sequence associated for $f(t)$	The First Expansion Theorem $T = \sum_{n \geq 0} \frac{[T\, p_n(z)]\vert_{z=0}}{n_\psi} Q(\partial_\psi)^n$ ∂_ψ - basic polynomial sequence $\{p_n\}_0^\infty$
$\exp\{y\overline{f}(t)\} = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} t^k$	$\exp_\psi\{xQ^{-1}(x)\} = \sum_{k \geq 0} \frac{p_k(y)}{k!} z^k$
The Sheffer Identity: $s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_n(y) s_{n-k}(x)$	The Sheffer ψ-Binomial Theorem: $s_n(x +_\psi y) = \sum_{k \geq 0} \binom{n}{k}_\psi s_k(x) q_{n-k}(y)$

Viskov	**Rota - oriented** (this note)
θ_ψ - the ψ-derivative $\theta_\psi\, x^n = \frac{\psi_{n-1}}{\psi_n} x^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi\, x^n = n_\psi\, x^{n-1}$

Viskov	**Rota - oriented** (this note)
$A_p\ (p=\{p_n\}_0^\infty)$ $A_p\, p_n = p_{n-1}$	Q $Q\, p_n = n_\psi p_{n-1}$
$B_p\ (p=\{p_n\}_0^\infty)$ $B_p\, p_n = (n+1)\, p_{n+1}$	$\hat{x}_Q$ $\hat{x}_Q\, p_n = \frac{n+1}{(n+1)_\psi} p_{n+1}$
$E_p^y\ (p=\{p_n\}_0^\infty)$ $E_p^y p_n(x) = \sum\limits_{k=0}^{n} p_{n-k}(x) p_k(y)$	$E^y\,(\partial_\psi) \equiv exp_\psi\{y\partial_\psi\}$ $E^y\,(\partial_\psi)\, p_n(x) =$ $= \sum\limits_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
$T-\varepsilon_p$-operator: $T\,A_p = A_p\,T$	E^y - shift operator: $E^y\varphi(x) = \varphi(x +_\psi y)$
$\forall_{y\in F}\ T E_p^y = E_p^y T$	T - ∂_ψ-shift invariant operator: $\forall_{\alpha\in F}\ [T, E^\alpha(\partial_\psi)] = 0$
Q - δ_ψ-operator: Q - ϵ_p-operator and $Qx = const \neq 0$	$Q(\partial_\psi)$ - ∂_ψ-delta-operator: $Q(\partial_\psi)$ - ∂_ψ-shift-invariant and $Q(\partial_\psi)(id) = const \neq 0$

Viskov	**Rota - oriented** (this note)
$\{p_n(x), n \geq 0\}$ - (Q,ψ)-basic polynomial sequence of the δ_ψ-operator Q	$\{p_n\}_{n\geq 0}$ -∂_ψ-basic polynomial sequence of the ∂_ψ-delta-operator $Q(\partial_\psi)$
ψ-binomiality property $\Psi_y s_n(x) =$ $= \sum_{m=0}^{n} \frac{\psi_n \psi_{n-m}}{\psi_n} s_m(x) p_{n-m}(y)$	ψ-binomiality property $E^y(\partial_\psi)p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)p_{n-k}(y)$
$T = \sum_{n\geq 0} \psi_n [VTp_n(x)] Q^n$ $T\Psi_y p(x) =$ $\sum_{n\geq 0} \psi_n s_n(y) Q^n STp(x)$	$T = \sum_{n\geq 0} \frac{[Tp_n(z)]\|_{z=0}}{n_\psi!} Q(\partial_\psi)^n$ $Tp(x +_\psi y) =$ $\sum_{k\geq 0} \frac{s_k(y)}{k_\psi!} Q(\partial_\psi)^k STp(x)$

Markowsky	**Rota - oriented**
L - the differential operator $L\, p_n = p_{n-1}$	Q $Q\, p_n = n_\psi p_{n-1}$
M $M\, p_n = p_{n+1}$	$\hat{x}_Q$ $\hat{x}_Q\, p_n = \frac{n+1}{(n+1)_\psi} p_{n+1}$
L_y $L_y\, p_n(x) =$ $= \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$	$E^y\,(Q) = \sum_{k\geq 0} \frac{p_k(y)}{k_\psi!} Q^k$ $E^y\,(Q)\; p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
E^a - shift-operator: $E^a\, f(x) = f(x+a)$	E^y - ∂_ψ-shift operator: $E^y \varphi(x) = \varphi(x +_\psi y)$
G - shift-invariant operator: $EG = GE$	T - ∂_ψ-shift invariant operator: $\forall_{\alpha \in F}\; [T, E(Q)] = 0$
G - delta-operator: G - shift-invariant and $Gx = const \neq 0$	$L = L(Q)$ - Q_ψ-delta operator: $[L, Q] = 0$ and $L(id) = const \neq 0$

Markowsky	Rota - oriented
$D_L(G)$ L - Pincherle derivative of G $D_L(G) = [G, M]$	$G' = [G(Q), \hat{x}_Q]$ Q - Pincherle derivative
$\{Q_0, Q_1, ...\}$ - basic family for differential operator L	$\{p_n\}_{n\geq 0}$ -ψ-basic polynomial sequence of the generalized difference operator Q
binomiality property $P_n(x+y) =$ $= \sum_{i=0}^{n} \binom{n}{i} P_i(x) P_{n-i}(y)$	Q - ψ-binomiality property $E^y(Q)p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_{\psi} p_k(x) p_{n-k}(y)$

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