Quasi-commutative cochains in algebraic topology

(After Max Karoubi)

We introduce a new type of "differential forms" on a simplicial set (= a space)over a basic commutative coherent ring k, for instance $\mathbb{Z}$, $\mathbb{F}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}$... They define a differential graded algebra, $\mathcal{D}^*(X)$, called quasi-commutative, which is in fact quasi-isomorphic to the usual algebra of cochains $C^*(X)$. The algebra $\mathcal{D}^*(X)$ determines (assuming some finiteness conditions) the homotopy type of X when $k = \mathbb{Z}$, a result linked with the work of Quillen and Sullivan in rational homotopy, in what case the differential algebras are genuine commutative algebras and not only quasi-commutative, as in our case.
Given two spaces X and Y, the cup-product is defined thanks to a map

$$m_{X,Y} : \mathcal{D}^*(X) \otimes \mathcal{D}^*(Y) \to \mathcal{D}^*(X \times Y)$$

We shall construct functorially a subcomplex $\mathcal{D}^*(X)\bar{\otimes}\mathcal{D}^*(Y) \subset \mathcal{D}^*(X) \otimes \mathcal{D}^*(Y)$ called the reduced tensor product, and verifying mainly the two following conditions :

- the preceding inclusion is a quasi-isomorphism (i.e. induces an isomorphism in cohomology)

-the following diagram (where the vertical arrows are induced by switching round X and Y) is commutative :

$$\begin{array}{ccc} \mathcal{D}^*(X)\bar{\otimes}\mathcal{D}^*(Y) & \xrightarrow{m_{X,Y}} & \mathcal{D}^*(X \times Y) \\ \downarrow & & \downarrow \\ \mathcal{D}^*(Y)\bar{\otimes}\mathcal{D}^*(X) & \xrightarrow{m_{Y,X}} & \mathcal{D}^*(Y \times X) \end{array} .$$

The main aim of the talk will be to present the construction of the functor D provided with its additional structure given by $m_{X,Y}$ and the reduced tensor product $\bar{\otimes}$.

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