Contact and equivalence of submanifolds of homogeneous spaces

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All manifolds and maps are assumed to be of class C^∞ unless otherwise stated.

Let G be a Lie group acting on the manifold N. Two submanifolds S, $\overline{S}$ of N, $S, \overline{S} \subset N$, are G-equivalent if there exists $g \in G$ such that $g.S = \overline{S}$. S and $\overline{S}$ have G-contact of order $k \geq 0$ at points $a \in S$ and $\overline{a} \in \overline{S}$ if there exists $g \in G$ such that $g.a = \overline{a}$ and $g.S$ and $\overline{S}$ have contact of order k at the point $\overline{a}$. S and $\overline{S}$ are locally G-equivalent at points a and $\overline{a}$ if there are open neighborhoods of a and $\overline{a}$ which are G-equivalent.

We shall consider the following problems:

(1) Let $\varphi : S \to \overline{S}$ be a diffeomorphism such that S and $\overline{S}$ have G-contact of order k at corresponding points $a \in S$ and $\varphi(x) \in \overline{S}$. Can we always choose k sufficiently high to ensure that S and $\overline{S}$ are G-equivalent?

(2) Under which conditions there exists $g \in G$ such that φ is the restriction to S of the map $L_g : x \in M \to g.x \in M$?

(3) Assume that G sets transitively on M. Give conditions on the contact elements of S for S to be a homogeneous space of a Lie subgroup of G.

Assuming regularity conditions on S and $\overline{S}$, the integer asked for in problem 1) always exists.

Let p be the dimension of S. G acts in a natural way in the manifold $C^{k,p}M$ of contact elements of order k and dimension p of M [ ]. For a point $a \in S$, let $C^k_a S$, G^k_a and d^k_{ra} be respectively the contact element of order k of S ate the point a, the isotropy subgroup of G at the point $C^k_a S \in G^{k,p}M$ and the dimension of G^k. Let $X = C^k_a S$ and denote by O^k_x and $h^k(x)$ respectively the orbit of $C^k_x S$ in $G^{k,p}M$ and the dimension of the vector space $T_X O^k_x \cap T_X C^k S$, where $C^k S$ is the submanifold of $G^{k,p}M$ of all contact elements of order k of S and $T_X O^k_x$ and $T_X C^k S$ are the tangent spaces of O^k_x and $C^k S$ at the point x.

For $k' \leq k$, $d^k(a) \leq d^{k'}(a)$ and $h^k(a) \leq h^{k'}(a)$. Hence, there exists an integer $k \geq 1$, such that $d^k(a) = d^{k-1}(a)$ and $h^k(a) = h^{k-1}(a)$. We say that a is a G-regular point of S if there exists $k \geq 1$ such that 1) $d^k(a) = d^{k-1}(a)$ and $h^k(a) = h^{k-1}(a)$, 2) $d^k(x)$ and $h^a(x)$ are constant for x in a neighborhood of a in S. If these conditions are satisfied we say that a is a k-regular point of S under the action of G. If a is a k-regular point of S thus $g.a$ is a k-regular point of $g.S$. The order of a is the least integer k satisfying the two conditions above.

Theorem 1 *Let $S, \overline{S} \subset M$ be two submanifolds of M of same dimension p. Let $a \in S$ and $\overline{a} \in \overline{S}$ be two points. Assume that $\overline{a}$ is a k-regular point of $\overline{S}$ and that there exists a continuous map $\varphi : V \to G$, defined in a neighborhood V of a in S such that $\varphi(a).a = \overline{a}$, $\varphi(x).x \in \overline{S}$ and $\varphi(x).C^k_x S = C^k_{\varphi(x)} \overline{S}_a$ for all $x \in V$. Then, there exist open neighborhoods W and $\overline{W}$ of a and $\overline{a}$ in S and $\overline{S}$ which are G-equivalent.*

The proof of theorem 1 rests on the theorem of uniqueness of solutions of partial differential systems of finite type [1], [2].

In theorems 2 and 3 we assume that G is a compact Lie group and H is a closed subgroup of G. Let L be the union of all G-orbits of $C^{k,p}M$ of type H that is, orbits whose isotropy subgroups are conjugate to H. Denote by N the quotient space of L by the orbits and by $\pi : L \to N$ the natural projection. It is known [3] that $(L, N, \pi, G/H, G)$ is a differentiable fiber bundle with structural group G.

Let $f : S \to \overline{S}$ be a diffeomorphism and such that S and $\overline{S}$ have G-contact of order $k \geq 1$ at corresponding points $x \in S$ and $\overline{x} = f(x) \in \overline{S}$ and let $a \in S$ and $\overline{a} \in f(a) \in \overline{S}$ be two points. Considering suitable cross sections of the fiber bundle $(L, N, \pi, G/H, G)$ one can prove the existence of a neighborhood V of a in S and of a differentiable map $\varphi : V \to G$ such that $\varphi(x).x = f(x)$ and $\varphi(x).C^k_x S = C^k_{\overline{x}}\overline{S}$. Hence, theorem 1 can be restated as follows:

Theorem 2 *Assume that there exists $k \geq 1$ and that*

Theorem 3 *Assume that S and $\overline{S}$ are connected and that there exists an integer $k \geq 1$ such that:*

Consider again the fiber bundle $(L, N, \pi, G/H, G)$. There exists a finite number of real valued differentiable functions $\tilde{\rho}_i$, $1 \leq i \leq r$, defined in L, such that two contact elements $X, \overline{X} \in L$ are in the same orbit of G if and only if $\tilde{\rho}_i(X) = \tilde{\rho}_i(\overline{X})$. Given a submanifold $S \subset M$ of dimension p, if the orbits of $C^k_x S$ are of type H for all $x \in S$ one can pull back the functions $\tilde{\rho}_i$ by the map $\sigma^k : x \in S \to C^k_x S \in L$. The functions $\rho_i = \tilde{\rho}_i \circ \sigma^k$ are called a complete set of G-invariants of order k of the submanifold S of M. Often the invariants can be defined in a natural way and have deep geometrical meaning as for instance, the curvature and torsion of curves in $\mathbb{R}^3$.

Assuming that the isotropy subgroups of $C^k_x S$ and $C^k_{\overline{x}}\overline{S}$ are of type H for all

$x \in S$ and $\overline{x} \in \overline{S}$, the invariants of order k, ρ_i and $\overline{\rho}_i$ are defined in S and $\overline{S}$. One can then restate theorems 2 and 3 replacing condition (3) by the following condition:

The condition $h^k(x) = 0$ in theorem 3 is equivalent to the statement that the rank of the differentials $d\overline{\rho}_i$, $1 \leq i \leq r$, is p at every point $\overline{x} \in \overline{S}$.

Assume now that G acts transitively on M and let S be the orbit of a Lie subgroup K of G. Then, $h^k(x) = p$ and C_x^k, $C_{x'}^k$ are conjugate subgroups for all $x, x' \in S$ and every $k \geq 0$. Hence, there exists $k \geq 1$ such that every $x \in S$ is a k-regular point of S.

Theorem 4 *A necessary and sufficient condition for a submanifold $S \subset M$ to be an open set of an orbit of a connected Lie subgroup K of G is that there exists $k \geq 1$ such that x is a k-regular point of S and $h^k(x) = p$ for all $x \in S$.*

If we assume moreover that G is compact, then a complete set ρ_i of invariants of order k is defined on S. Clearly $h^k(x) = p$ for every $p \in S$ if and only if the functions ρ_i are constant on S. Hence, the following corollary to theorem 4 holds:

Theorem 5 *Let $S \subset M$ be a connected submanifold and assume G compact. Assume also that there exists $k \geq 1$ such that every point of S is k-regular and that the isotropy groups G_x^k are conjugate in G. Then, S is an open set of an orbit of a connected Lie subgroup of G if and only if the invariants of order k are constant on S.*

Bibliographie

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