IN CHARLES EHRESMANN'S FOOTSTEPS: FROM GROUP GEOMETRIES TO GROUPOID GEOMETRIES.

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In this lecture we intend to focus on one of the turning points in Charles Ehresmann's work: we mean, when dealing with the theory of principal bundles, the introduction, parallel to that of the *structural group*, of the *structural groupoid*. (Note that nowadays these are often known too as gauge group and gauge groupoid, rather conveniently though ambiguously, since these terms possess also quite different meanings, according to the authors). The actions, on the bundle, of the group and the groupoid are commuting.

Together with the study of the category of jets (and more specifically the groupoid of invertible jets), this led him to the basic idea (rediscovered much later by various authors) of considering (small) internal groupoids (more generally categories) in various (large) categories, originally and basically the category of (morphisms between) manifolds. In particular this leads to an inductive definition of multiple (smooth) groupoids. This unifying concept involves a very far-reaching intertwining between algebra and geometry.

He also stressed the interest of choosing the source and target maps and/or the anchor map (or transitor) in suitable subclasses.We shall show it is convenient to impose to these subclasses some suitable stabibity properties, and propose various choices unifying various theories. We point out that the stabiblity properties we need are very easily satisfied in the topos setting, but one gets much wider ranging theories (especially when aiming at applications to Differential Geometry) when not demanding the structuring category to be a topos.

We note that the concepts of gauge group and groupoid have a strong intuitive (geometrical or physical) interpretation. When thinking the elements of the principal bundle P as "events" or "observations", the fibres, which are also the elements of the base or the objects of the groupoid, may be viewed as "observers".

In that respect, the unique object of the group plays the role of a "universal observer", and the group itself of an "absolute" gauge reference. On the other hand the groupoid allows direct comparisons between the various observers, without using the medium of the absolute observer. The comparison between these two "points of view" (absolute and relative) is realized by means of the projections of the bundle P onto the bases of the group and of the groupoid (the former being a singleton), and by the basic fact that the group and the groupoid induce on P the "same" groupoid (more precisely isomorphic groupoids). From a purely algebraic point of view, this describes a Morita equivalence between the structural group and groupoid. But in Ehresmann's internal setting, this acquires a lot of different meanings depending on the above-mentioned various choices.

This groupoid (which is the core of the structure) inherits from the two projections a very rich extra structure, which may be recognized as a particular instance of a very special and interesting structure of smooth double groupoid. In the purely algebraic context, this structure has been described in the literature under various names and equivalent ways: rule of three, affinoid (A. Weinstein), pregroupoid (A. Kock).

Now it turns out that all the essential features of this description are still valid when replacing the structural group by a groupoid, and the situation becomes perfectly symmetrical (up to isomorphism). In the purely algebraic (hence also topos) setting, this symmetric situation has been described by A. Kock, in the language of torsors and bitorsors.

In the "physical" interpretation, we can say that we have now two classes of observers, and a comparison between two "conjugate points of view". We also note that it is no longer demanded the groupoids to be transitive (then certain pairs of observers cannot compare their observations), and also the rank of the anchor map to be constant (then the isotropy groups are allowed to vary, and this may be thought as symmetry breakings or changes of phase).

We shall give various examples of this unifying (purely diagrammatic) situation (and more general ones), which encompasses the construction of associated bundles as well, the realization of a non abelian cocycle, including Haefliger cocycles, the construction of the holonomy groupoids of foliations, and the Palais globalization of a local action law.

By lack of time, we shall not describe the corresponding infinitesimal situation, which of course was Ehresmann's main motivation for introducing the structural groupoid, in order to understand the meaning of "infinitesimally connecting" the fibres. This will be tackled by other lecturers at the present Conference.

Our conclusion will be that the transition, from the Kleinian conception of Geometry as the study of group actions, to the Ehresmannian enlarged point of view, consisting in considering groupoid actions, involves a conceptual revolution which parallels the physical revolution from the classical concept of an absolute universe to the modern visions about our physical space.

It might well be that this revolution is not enough well understood presently, and is liable to come out into unexpected developments.

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