THE NATURE OF FIBRATIONS AND OF BUNDLES

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1. ABSTRACT

The notion of fibration, as a map that verifies the homotopy lifting property — say, for polytopes — was pointed out by Ehresmann in the early 40's. In this talk, I shall first present a characterization, in terms of three properties.

-the first one (homotopical submersion) is local at the source, and thus verified by submersions, as well as by fibrations;

-the two other ones (triviality of emerging cycles and of vanishing cycles of every dimension) express the isomorphism of the homotopy groups of the neighboring fibres.

If we turn to the category of open manifolds of finite dimension and smooth maps, this allows a caracterization of bundles (maps which are trivial over an open covering of the base) among submersions (maps whose differential at each point is onto), in terms of the homotopy type of the space of embeddings of (large) compact domains in the fibres.

In particular, consider a submersion $E \to B$ where E, B are open manifolds; and assume that it is a fibration, i.e. it verifies the homotopy lifting property. In general, it may not be a bundle. For example, there is a submersion-fibration over $B = \mathbf{R}$ with all fibres $\mathbf{R}^3$ but one which is the Whitehead manifold, a contractible open 3-manifold not homeomorphic to $\mathbf{R}^3$.

We show that every submersion-fibration is necessarily a bundle under any of the following hypotheses on the topology of the fibres:

The dimension of the fibres is 2 .

Each fibre is diffeomorphic to $\mathbf{R}^p$.

Each fibre is topologically finite, of dimension at least 5, and its boundary at infinity is simply connected.

Also, there is a stabilization result: every submersion-fibration becomes a bundle if we multiply the fibres by some $\mathbf{R}^N$.

Finally, we give applications to foliations.