HOMOTOPY TYPES OF STABILIZERS AND ORBITS OF MORSE MAPPINGS OF SURFACES

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Let M be a smooth compact surface (oriented or not, with boundary or without it) and P either $\mathbb{R}$ or S^1. The group $\mathcal{D}(M)$ of diffeomorphisms of M naturally acts on $C^\infty(M,P)$ by the following rule: if $h \in \mathcal{D}(M)$ and $f \in C^\infty(M,P)$, then $h \cdot f = f \circ h$.

For $f \in C^\infty(M,P)$ let $\mathcal{S}(f) = \{h \in \mathcal{D}(M) \,|\, f \circ h = f\}$ be the *stabilizer* and $\mathcal{O}(f) = \{f \circ h \,|\, h \in \mathcal{D}(M)\}$ be the *orbit* of f under this action.

Let Σ_f be the set of critical points of f and $\mathcal{D}(M, \Sigma_f)$ the group of diffeomorphisms h of M such that $h(\Sigma_f) = \Sigma_f$. Then the stabilizer $\mathcal{S}(f, \Sigma_f)$ and the orbit $\mathcal{O}(f, \Sigma_f)$ of f under the restriction of the above action to $\mathcal{D}(M, \Sigma_f)$ are well defined. Evidently, $\mathcal{S}(f) \subset \mathcal{D}(M, \Sigma_f)$, whence $\mathcal{S}(f, \Sigma_f) = \mathcal{S}(f)$.

Let $\mathcal{S}_{\mathrm{id}}(f)$ be the identity path-component of $\mathcal{S}(f)$ and $\mathcal{O}_f(f)$ and $\mathcal{O}_f(f, \Sigma_f)$ the connected components of $\mathcal{O}(f)$ and $\mathcal{O}(f, \Sigma_f)$ (resp.) in the corresponding compact-open topologies. We endow $\mathcal{S}_{\mathrm{id}}(f)$, $\mathcal{O}_f(f)$, and $\mathcal{O}_f(f, \Sigma_f)$ with C^∞-topologies.

A function $f : M \to P$ will be called *Morse* if $\Sigma_f \subset \mathrm{Int}M$, all critical points of f are non-degenerated, and f is constant on the connected components of ∂M. A Morse mapping f is *generic* if every level-set of f contains at most one critical point; f is *simple* if every critical component of a level-set of f contains precisely one critical point. Evidently, every generic Morse mapping is simple.

Let $f : M \to P$ be a Morse mapping. We fix some orientation of P. Then the index of a non-degenerated critical point of f is well-defined. Denote by c_i, $(i = 0, 1, 2)$ the number of critical points of f of index i.

Theorem 1. *If either $c_1 > 0$ or M is non-orientable, then $\mathcal{S}_{\mathrm{id}}(f)$ is contractible. Otherwise, $\mathcal{S}_{\mathrm{id}}(f)$ is homotopy equivalent to S^1.*

Theorem 2. *Suppose that $c_1 > 0$. Then*

(1) $\mathcal{O}_f(f, \Sigma_f)$ is contractible;

(2) $\pi_i \mathcal{O}_f(f) \approx \pi_i M$ for $i \geq 3$ and $\pi_2 \mathcal{O}_f(f) = 0$. In particular, $\mathcal{O}_f(f)$ is aspherical provided so is M. Moreover, $\pi_1 \mathcal{O}_f(f)$ is included in the following exact sequence

$$0 \to \pi_1 \mathcal{D}(M) \oplus \mathbb{Z}^k \to \pi_1 \mathcal{O}_f(f) \to G \to 0, \qquad (1)$$

where G is a finite group and $k \geq 0$. If f is simple, then for the surfaces presented in Table 1, the number k is determined only by the number of critical points of f.

TABLE 1

M	k
$M = S^2, D^2, S^1 \times I, T^2, P^2$ *with holes*	$c_1 - 1$
M *is orientable and differs from the surfaces above*	$c_0 + c_2$

(3) Suppose that f is generic. Then the group G in Eq. (1) *is trivial, whence $\pi_1\mathcal{O}(f) \approx \pi_1\mathcal{D}(M) \oplus \mathbb{Z}^k$. In particular, $\pi_1\mathcal{O}(f)$ is abelian. The homotopy type of $\mathcal{O}_f(f)$ is given in Table 2.*

TABLE 2

Surface M	*Homotopy type of $\mathcal{O}_f(f)$*
S^2, P^2	$SO(3) \times (S^1)^{c_1-1}$
D^2, $S^1 \times I$, *Möbius band Mo*	$(S^1)^{c_1}$
T^2	$(S^1)^{c_1+1}$
Klein bottle K	$(S^1)^{k+1}$
other cases	$(S^1)^k$

Theorem 3. *If $c_1 = 0$, then f can be represented in the following form*

$$f = p \circ \widetilde{f} : M \xrightarrow{\widetilde{f}} \widetilde{P} \xrightarrow{p} P,$$

where $\widetilde{f}$ is one of the mappings shown in Table 3, $\widetilde{P}$ is either $\mathbb{R}$ or S^1, and p is either a covering map or a diffeomorphism. In this case the homotopy types of $\mathcal{O}_f(f)$ and $\mathcal{O}_f(f, \Sigma_f)$ depend only on $\widetilde{f}$ and are given in Table 3.

Here K is represented as the factor space of the 2-torus $T^2 \approx S^1 \times S^1$ by the involution $(x, y) \mapsto (x + \pi/2, -y)$ and $*$ means contractibility.

TABLE 3

Type	$\widetilde{f} : M \to \widetilde{P}$		c_0	c_1	c_2	$\mathcal{O}_f(f)$	$\mathcal{O}_f(f, \Sigma_f)$
(A)	$S^2 \to \mathbb{R}$	$\widetilde{f}(x, y, z) = z$	1	0	1	S^2	$*$
(B)	$D^2 \to \mathbb{R}$	$\widetilde{f}(x, y) = x^2 + y^2$	1/0	0	0/1	$*$	
(C)	$S^1 \times I \to \mathbb{R}$	$\widetilde{f}(\phi, t) = t$	0	0	0	$*$	
(D)	$T^2 \to S^1$	$\widetilde{f}(x, y) = x$,	0	0	0	S^1	
(E)	$K \to S^1$	$\widetilde{f}(\{x\}, \{y\}) = \{2x\}$	0	0	0	S^1	

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