A solvability of nonlinear equations with noninvertible linear part
Lectures in Ioannina

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1 Introduction

The most of nonlinear equations can be represented in a form

$$Lx = N(x) \tag{1.1}$$

where L is a linear operator on Banach spaces and N is a nonlinear map. We shall study the existence of a solution for (1.1) and the lower bound for the number of solutions (the last is referred as multiplicity results). Obviously, the linear part L is not determined uniquely but the study of the equation is simpler if one uses an appropriate linear part. For example, the nonlinear elliptic Dirichlet problem

$$\Delta u = f(u), \qquad u|\partial\Omega = 0 \tag{1.2}$$

usually has the linear part Δ – the Laplace operator considered on appropriate Sobolev or Hölder spaces – and the nonlinear part being the superposition operator $u \to f \circ u$, but if the function f is asymptotically linear:

$$\lim_{u\to\pm\infty} \frac{f(u)}{u} = \lambda, \tag{1.3}$$

one can use the linear part $u \to \Delta u - \lambda u$ and the nonlinear one $u \to f \circ u - \lambda u =: \tilde{f} \circ u$ with $\tilde{f}$ being sublinear

$$\lim_{u\to\pm\infty} \frac{\tilde{f}(u)}{u} = 0.$$

The second approach is much simpler if one uses topological methods, fixed point theory and so on.

Thus, let us consider equation (1.1) with $L : X \supset \text{dom } L \to Z$, $N : X \to Z$, where X, Z are Banach spaces, dom L is a linear subspace (usually dense) and N is a continuous nonlinear operator. We do not assume that the linear operator L is continuous. The main question is whether there exists a solution.

If L is invertible and its inverse is continuous (as it is in applications), then we have the equivalent equation

$$x = L^{-1}N(x)$$

and the existence of a solution follows from one of fixed point theorems. For example, in the case of the above mentioned elliptic problem (1.2), if λ given by (1.3) is not an eigenvalue, then $\Delta - \lambda I$ is invertible on appropriate Sobolev spaces and its inverse is compact. Hence the existence of a solution is a consequence of the Schauder Fixed Point Theorem on a sufficiently large ball.

One finds the first difficulty when L is not invertible but only a Fredholm operator of index zero, i.e. the kernel of L is nontrivial and finite dimensional, the image of L has a finite codimension in Z and

$$\dim\ker L = \text{codim}_Z \text{ im } L.$$

For our elliptic problem (1.2), this is the case if λ is an eigenvalue. The abstract equation (1.1) can be reduced to the system

$$\text{(1.4)} \qquad \begin{cases} x = KN(x) + v \\ N(x) \in \text{im } L, \end{cases}$$

where v is an arbitrary element of the kernel and K is a right inverse of L :

$$LK = I_{\text{im}L}.$$

This case can be examined by using Mawhin's coincidence degree [27, 53] or by the perturbation method [63, 64, 65, 66, 67] if K is compact. If N is sublinear

$$\text{(1.5)} \qquad \lim_{||x|| \to \infty} \frac{N(x)}{||x||} = 0$$

then the existence of a solution depends on the behaviour of $N| \ker L$.

To be more precise: let $u_1, \dots, u_m$ be linear bounded functionals on Z being linearly independent such that

$$\text{im } L = \bigcap_{j=1}^{m} \ker u_j$$

– they always exist if codim im $L = m$ – and let $v_1 \dots, v_m$ constitute the basis of the kernel of L.. Then, if for any $j \in \{1, \dots, m\}$, $d_1, \dots, d_{j-1}, d_{j+1}, \dots, d_m \in \mathbb{R}$

$$d_j u_j (N(\sum_{i=1}^{m} d_i v_i)) \leq 0$$

for sufficiently large d_j, then equation (1.1) has a solution.

The above result is, in fact, the abstract version of the famous theorem from the first paper on boundary value problems at resonance by Landesman and Lazer [46], thus we call the assumption as Landesman-Lazer type condition (see [63, 65, 67]). For the example (1.2) and λ being the first eigenvalue, we have $m = 1$, v_1 is a positive function on Ω, $u_1(z) = \int_\Omega v_1 \cdot z$ and the solvability follows from the assumption:

— two limits $\lim_{u \to \pm\infty} f(u)$ have opposite signs.

The method can be applied to higher eigenvalues and the nonlinearities with a linear growth,

$$|f(u)| \leq a|u| + b,$$

but assumptions are more complicated (comp. [64]). The nonlinearity f can also depend on the independent variable x and then Landesman-Lazer condition has the usual integral form. We shall see it below.

The method works for ODEs as well; typical problems are

$$x' = f(t, x), \quad x(0) = x(T), \quad \text{−periodic problem,}$$

$$x'' + n^2 x = f(t, x, x'), \quad x(0) = 0 = x(\pi), \quad \text{−Dirichlet problem}$$

and by another kind of problems (functional-differential equations, impulsive differential equations, nonlocal equations).

2 Coincidence degree

Let $L : X \supset Y \to Z$ be a linear operator and $N : X \to X$ a nonlinear one. We suppose that L is Fredholm operator of index zero:

$$\dim \ker L = \dim Z/\mathrm{im}\ L < \infty,$$

P – a linear continuous projector on $\ker L$, Q – a projector along $\mathrm{im}\, L$ onto its topological complement. Denote by K_P the inverse operator to

$$L|\ker P \cap Y : \ker P \cap Y \to \mathrm{im}\ L.$$

We assume that N is L-completely continuous, i.e. QN and $K_P(I-Q)N$ are completely continuous. In applications usually K_P is completely continuous and N is continuous and maps bounded sets on bounded ones.

Theorem 2.1. (Mawhin Continuation Principle) *Let Ω be a bounded open set in X, all equations $Lx = \lambda N(x)$ have no solutions $x \in \partial\Omega \cap Y$ for $\lambda \in (0, 1]$ and the Brouwer degree*

$$\deg(JQN|\ker L, \Omega \cap \ker L, 0) \neq 0,$$

where $J : \mathrm{im}\ Q \to \ker L$ is an arbitrary isomorphism, then equation $Lx = N(x)$ has a solution in $\overline{\Omega}$.

The Brouwer degree from this theorem is called the coincidence degree of the pair (L, N) and is denoted by $\deg_M((L, N), \Omega)$. Its properties are similar as properties of the Leray-Schauder degree and can be found in [53]. This is a generalization of this degree:

$$\deg_{LS}(I - N, \Omega, 0) = \deg_M((L, N), \Omega).$$

The coincidence degree has found a lot of applications in proving the existence of a solution to BVPs. A crucial question is (as for invertible linear part and the L-S. degree) to get a priori estimates for a homotopical family of problems, but now, one needs some extra assumptions to guarantee the finite-dimensional degree does not vanish. This assumptions are especially simple when the kernel $\ker L$ is 1-dimensional. Then $\Omega \cap \ker L$ is an interval and QN should takes opposite sign at its both ends. This is again the Landesman-Lazer condition.

An interesting generalization of the theory can be found in the paper by L. Nirenberg [59], where an index of L can be arbitrary positive number p. However, in this case, one needs the homotopy invariant of mappings $S^{n+p} \to S^n$: Nirenberg applied its stable homotopy class. Since the stable homotopy groups are complicated and there is no correspondance between them and the group $\mathbb{Z}$, for instance, the Nirenberg's theory is difficult for application.

We can find similar approaches in the literature, we refer to [22].

3 Perturbation method

Let as above X and Z be Banach spaces, Y a linear subspace of X and L strongly continuous family of operators $Y \to Z$ depending on real parameter λ

Suppose that $L(\lambda)$ are invertible for $\lambda \neq \lambda_0$ and $L(\lambda_0)$ is Fredholm operator with nontrivial kernel. Its index is then 0. Moreover, suppose the family of inverse operators $L(\lambda)^{-1} =: G(\lambda)$ can be represented in the form:

$$G(\lambda) = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), \cdot\rangle w_j(\lambda), \tag{3.1}$$

where $G_0(\lambda) \in L(Z, X)$ are completely continuous, $u_j(\lambda) \in Z^*$ (space of all continuous linear functionals on Z), $w_j(\lambda) \in Y$ are continuous functions of λ having continuous extensions for $\lambda = \lambda_0$, and $c_j(\lambda) \in \mathbb{R}$ provided that

$$\lim_{\lambda\to\lambda_0} |c_j(\lambda)| = \infty,$$

for all $j = 1, \dots, n$. Let moreover, $w_j(\lambda)$, $j = 1, \dots, n$, be linearly independent vectors that span $\ker L(\lambda_0)$ for $\lambda = \lambda_0$,

$$\operatorname{im} L(\lambda_0) = \bigcap_{j=1}^{n} \ker u_j(\lambda_0).$$

Then vectors $w_j(\lambda)$, $j = 1, \dots, n$ and covectors $u_j(\lambda)$, $j = 1, \dots, n$, are linearly independent for λ sufficiently close to λ_0. At last assume operator $G_0(\lambda_0)$ is a right inverse to $L(\lambda_0)$, i.e.

$$L(\lambda_0)G_0(\lambda_0)z = z, \qquad z \in \operatorname{im} L(\lambda_0).$$

Let $N : X \to Z$ be a nonlinear continuous operator mapping bounded sets into bounded ones. The equation

$$L(\lambda_0)y = N(x). \tag{3.2}$$

is equivalent to the system

$$\begin{array}{l} x = G_0(\lambda_0)N(x) + \sum_{j=1}^{n} d_j w_j(\lambda_0), \\ \langle u_j(\lambda_0), N(x)\rangle = 0, \qquad j = 1, \dots, n, \end{array} \tag{3.3}$$

Real constants $d_1, \dots, d_n$ are arbitrary.

There is natural question, when the family $L(\lambda)$ admits representation 3.1. We present a simple sufficient condition. Let $w_1, \dots, w_n$ constitutes a basis of $\ker L(\lambda_0)$. Assume that there exist limits

$$\lim_{\lambda\to\lambda_0} ||L(\lambda)w_j||^{-1}L(\lambda)w_j =: h_j, \qquad j = 1, \dots, n \tag{3.4}$$

and

$$\operatorname{im} L(\lambda_0) \oplus \operatorname{Lin}\{h_1, \dots, h_n\} = Z. \tag{3.5}$$

We construct 3.1. Let X_0 be a topological complement of $X_1 = \ker L(\lambda_0)$. For $\lambda \neq \lambda_0$ define $u_j(\lambda) \in Z^*$, $j = 1, \dots, n$, by the formula

$$\begin{aligned} \langle u_j(\lambda), L(\lambda)w_i\rangle &= \delta_{ij} L(\lambda) w_i, \\ u_j(\lambda) | L(\lambda) X_0 &= 0. \end{aligned}$$

These functionals depend continuously on λ and have limits as $\lambda \to \lambda_0$ denoted by $u_j(\lambda_0)$ such that

$$\langle u_j(\lambda_0), h_i\rangle = \delta_{ij}, \qquad u_j(\lambda_0) | \operatorname{im} L(\lambda_0) = 0.$$

Now, put $P_1(\lambda)$ and $P_0(\lambda)$ for complementary projectors of Z onto $L(\lambda)X_1$ and $L(\lambda)X_0$ respectively. From 3.4 they have continuous extensions for $\lambda = \lambda_0$ being projectors onto $\operatorname{Lin}\{h_1, \dots, h_n\}$ and $\operatorname{im} L(\lambda_0)$. Thus, we can set

$$G_0(\lambda) = \begin{cases} G(\lambda)P_0(\lambda) & \text{dla} \ \ \lambda \neq \lambda_0, \\ (L(\lambda_0)|X_0)^{-1} P_0(\lambda_0) & \text{dla} \ \ \lambda = \lambda_0, \end{cases}$$

and the second summand of $G(\lambda) = G_0(\lambda) + G(\lambda)P_1(\lambda)$ has the form

$$G(\lambda)P_1(\lambda) z = \sum_{j=1}^{n} ||L(\lambda)w_j||^{-1} \langle u_j(\lambda), z\rangle \, w_j,$$

and we get the needed representation with $c_j(\lambda) = ||L(\lambda)w_j||^{-1}$.

Condition 3.4 holds, for example, for $L(\lambda) = L(0) + \lambda I$ and one-sided limits $\lambda > 0$ or $\lambda < 0$. Condition 3.5 means then

$$\ker L(0) \cap \operatorname{im} L(0) = \{0\}.$$

After [65] we repeat the main continuation result for our perturbation method which corresponds the Mawhin Continuation Principle. Under the above assumptions, we have a division of the space X into the topological sum

$$X = \tilde{X}_\lambda \oplus \overline{X}_\lambda$$

where $\overline{X}_\lambda = \operatorname{Lin}\{w_j(\lambda) : j = 1, \dots, n\}$.

Theorem 3.1. *Suppose there exists a bounded open set $\Omega \subset X \times (\lambda_0, \lambda_1]$ such that, for any solution of the system*

$$\begin{aligned} \tilde{x}_\lambda &= (\lambda - \lambda_1)(\lambda_0 - \lambda_1)^{-1} G_0(\lambda) N(x), \\ d_j &= c_j(\lambda)\langle u_j(\lambda), N(x)\rangle, \qquad j = 1, \dots, n, \end{aligned} \tag{3.6}$$

where $x = \tilde{x}_\lambda + \sum d_j w_j(\lambda)$*, we have* $(x, \lambda) \notin \partial\Omega$*. If* $g = (g_1, \dots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$ *given by*

$$g_j(d_1, \dots, d_n) = d_j - c_j(\lambda_1)\langle u_j(\lambda_1), N(\sum_i d_i w_i(\lambda_1))\rangle$$

satisfies

$$\deg(g, U, 0) \neq 0 \qquad \textit{(Brouwer degree)},$$

where $U = \{d \in \mathbb{R}^n : (\sum d_i w_i(\lambda_1), \lambda_1) \in \Omega\}$*, then equation* $L(\lambda_0)x = N(x)$ *has a solution* x *such that* $(x, \lambda_0) \in \overline{\Omega}$*.*

Proof. Define the homotopy $H : X \times (\lambda_0, \lambda_1] \to X$ by

$$H(x, \lambda) = \frac{\lambda - \lambda_1}{\lambda_0 - \lambda_1} G_0(\lambda) N(x) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), N(x)\rangle w_j(\lambda).$$

From the assumptions, it has no fixed points in the boundary $\partial\Omega$, which means that the Leray-Schauder degree

$$\deg_{LS}(I - H(\cdot, \lambda), \Omega_\lambda, 0),$$

with $\Omega_\lambda = \{x \in X : (x, \lambda) \in \Omega\}$ does not depend on λ. But $H(\cdot, \lambda_1)$ is a finite dimensional mapping, thus choosing basis $w_j(\lambda_1)$, $j = 1, \dots, n$, in the space $\overline{X}_{\lambda_1}$ we obtain

$$\deg_{LS}(I - H(\cdot, \lambda_1), \Omega_{\lambda_1}, 0) = \deg(g, U, 0) \neq 0.$$

It follows that, for any λ, there exists a solution of the equation $x = H(x, \lambda)$ w Ω_λ.

Take a sequence $\lambda_k \to \lambda_0$ and denote by x_k the last solution, $k \in \mathbb{N}$. This sequence is bounded. Due to the complete continuity of G_0, one can choose convergent sequences (for simplicity, we do not use subindices)

$$\begin{aligned} &G_0(\lambda_k) N(x_k) \to x_0, \\ &c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle \to d_j, \quad j = 1, \dots, n. \end{aligned}$$

Thus $x_k \to x_0 + \sum d_j w_j(\lambda_0) =: x$. Since $|c_j(\lambda_k)| \to \infty$ for all j, we get

$$\langle u_j(\lambda_0), N(x)\rangle = 0,$$

and by the continuity of G_0 i N we have

$$x_0 = G_0(\lambda_0) N(x).$$

Hence, x satisfies system 3.3 and it is the needed solution. □

Now, we shall prove some results, where a priori estimates are a consequence of the bounded growth. Without loss of generality we can assume that $c_j(\lambda) > 0$.

Theorem 3.2. *Suppose that the nonlinear term N is sublinear, i.e.*

$$\lim_{||x||\to\infty} ||N(x)||/||x|| = 0,$$

and for any sequence (x_k) with the properties $||x_k|| \to \infty$ and $||x_k||^{-1}x_k \to \sum d_j w_j(\lambda_0)$, where $(d_1, \dots, d_n) \in \mathbb{R}^n$, there exists $j \in \{1, \dots, n\}$ such that $d_j \neq 0$ and

$$d_j\langle u_j(\lambda_0), N(x_k)\rangle \leq 0 \tag{3.7}$$

for sufficiently large $k \in \mathbb{N}$. Then our equation has a solution.

Proof. Take $\lambda_k \to \lambda_0$, $\lambda_k \neq \lambda_0$, and consider the equation $L(\lambda_k)y_k = N(x_k)$ or equivalently

$$x_k = G(\lambda_k)N(x_k).$$

For any fixed $k \in \mathbb{N}$ there exists $R_k > 0$ such that

$$||N(x)||/||x|| \leq ||G(\lambda_k)||^{-1}$$

for $||x|| \geq R_k$, and the mapping $G(\lambda_k)N$ maps the boundary of the ball centered at 0 with radius R_k into this ball and, due to Rothe Fixed Point Theorem, the perturbed equation has a solution. If the sequence (x_k) of these solutions is bounded, then as in the proof of the previous theorem one can pass to a convergent subsequence and its limit is a sought solution.

Suppose then, this sequence is unbounded, hence (passing to a subsequence) $||x_k|| \to \infty$. Thus

$$\begin{aligned} ||x_k||^{-1}x_k &= G_0(\lambda_k)(||x_k||^{-1}N(x_k)) \\ &\quad + \textstyle\sum_j ||x_k||^{-1}c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle w_j(\lambda_k) \end{aligned}$$

and by the sublinearity of N, the first summand tends to 0. Repeating arguments from the proof of the previous theorem once more, we find convergent subsequences

$$||x_k||^{-1}c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle \to d_j, j = 1, \dots, n.$$

Hence

$$||x_k||^{-1}x_k \to \sum_{j=1}^{n} d_j w_j(\lambda_0),$$

so if j is such that $d_j \neq 0$ (all d_j's cannot be 0), then for sufficiently large k, the sign of $\langle u_j(\lambda_0), N(x_k)\rangle$ is the same as the sign of d_j. This contradicts assumption 3.7 and therefore the unboundedness of the sequence (x_k) is excluded.. □

This theorem is proved in [64], and in [65] one can find its another proof. Inequalities 3.7 is called the Landesman-Lazer type condition and, as we shall see below, it correspods to the assumption from [46]. The last theorem has a simple corrolary.

Theorem 3.3. *Assume that for any sequence $(x_k) \subset X$ such that $||x_k|| \to \infty$ and $||x_k||^{-1}x_k \to \sum d_j w_j(\lambda_0)$, there exists the limit $\lim_{k\to\infty} N(x_k)$ depending only on $(d_1, \dots, d_n) \in \mathbb{R}^n \setminus \{0\}$ which will be denoted as N_d. Then the following condition is sufficient for the solvability of our equation:*

for any $d \in \mathbb{R}^n$ such that $||\sum d_j w_j(\lambda_0)|| = 1$, there exists $j \in \{1, \dots, n\}$ with the property

$$d_j \langle u_j(\lambda_0), N_d) \rangle < 0.$$

If $n = 1$, then it suffices

$$\langle u_1(\lambda_0), N_1) \rangle \cdot \langle u_1(\lambda_0), N_{-1}) \rangle < 0.$$

We shall study the case when the nonlinear term has a linear growth:

$$\gamma := \limsup_{||x||\to\infty} ||N(x)||/||x|| \in (0, \infty).$$

After [64] we state

Theorem 3.4. *Suppose that $\gamma||G_0(\lambda_0)|| < 1$ and there exist constants $R > 0$ and*

$$\sigma > \gamma||G_0(\lambda_0)||/(1 - \gamma||G_0(\lambda_0)||)$$

such that, for any j and $|d_j| \geq R$, $|d_i| \leq |d_j|$ for $i \neq j$, $\lambda \in (\lambda_0, \lambda_1]$, $\tilde{x}_\lambda \in \tilde{X}_\lambda$ with the property $||\tilde{x}_\lambda|| \leq \sigma||\sum d_i w_i(\lambda)||$, we have

$$d_j \langle u_j(\lambda), N(\tilde{x}_\lambda + \sum d_i w_i(\lambda)) \rangle \leq 0. \tag{3.8}$$

Then the equation has a solution.

Proof. Taking a less subinterval if it is necessary, we can find $\varepsilon > 0$, such that

$$\sigma \geq (\gamma + \varepsilon)||G_0(\lambda_0)||/(1 - (\gamma + \varepsilon)||G_0(\lambda_0)||)$$

for $\lambda \in (\lambda_0, \lambda_1]$, next $R_1 > 0$ so large that

$$||N(x)|| \leq (\gamma + \varepsilon)||x||$$

for $||x|| \geq R_1$. Then choose $R_2 > R$ and

$$R_3 > \sigma \sup\{||\sum d_j w_j(\lambda)|| : d \in \partial(-R_2, R_2)^n\}$$

with the property: if $d \in \partial(-R_2, R_2)^n$ or $||\tilde{x}_\lambda|| = R_3$, $\tilde{x}_\lambda \in \tilde{X}_\lambda$, then $||\tilde{x}_\lambda + \sum d_j w_j(\lambda)|| \geq R_1$. Consider the set $\Omega \subset X \times (\lambda_0, \lambda_1]$ being the set of pairs $(\tilde{x}_\lambda + \sum d_j w_j(\lambda), \lambda)$ such that $||\tilde{x}_\lambda|| < R_3$ and $|d_j| < R_2$ for $j = 1, \dots, n$. We can apply Theorem 3.1, since $\partial\Omega$ includes points for which $||\tilde{x}_\lambda|| = R_3$ or one of numbers $|d_j|$ equals R_2, and the remaining ones satisfy inequalities $|d_i| \leq |d_j|$. Moreover, then $||x|| \geq R_1$ and we can use the assumed inequality. If such a point x has satisfied the system from Theorem 3.1, then

$$||\tilde{x}_\lambda|| \leq (\gamma + \varepsilon)||G_0(\lambda_0)||(||\tilde{x}_\lambda|| + ||\sum d_i w_i(\lambda)||),$$

what implies

$$||\tilde{x}_\lambda|| \leq \sigma||\sum d_i w_i(\lambda)|| < R_3.$$

Then we have $|d_j| = R_2 \geq |d_i|$, $i \neq j$, and $d_j = c_j(\lambda)\langle u_j(\lambda), N(x)\rangle$, which contradicts inequality 3.8.

It remains to calculate the degree of the mapping g from Theorem 3.1. We apply the homotopy $h = (h_1, \dots, h_n) : \langle -R_2, R_2\rangle^n \times \langle 0, 1\rangle \to \mathbb{R}^n$ given by the formula

$$h_j(d_1, \dots, d_n; t) = d_j - tc_j(\lambda_1)\langle u_j(\lambda_1), N(\sum d_i w_i(\lambda_1))\rangle,$$

that links g with the identity mapping I and does not take the value 0 on the boundary $\langle -R_2, R_2\rangle^n$. Hence $\deg(g, (-R_2, R_2)^n, 0) = 1$ and the assertion follows from Theorem 3.1. $\square$

In [65] one can find an existence result for the case when the nonlinearity is superlinear. The above theorems generalize results of Kannan, McKenna [39], Šeda [80] and many others.

It is interesting that conditions 3.7 and 3.8 are close to necessary ones if the limits from Theorem 3.3 exist. We shall formulate the result only for the case $n = 1$.

Theorem 3.5. *Suppose that there exist limits*

$$\lim_{d\to\pm\infty} N(dw_1(\lambda_0)) =: N_\pm.$$

If, for any $x \in X$

$$\langle u_1(\lambda_0), N(x)\rangle \in (\langle u_1(\lambda_0), N_+\rangle, \langle u_1(\lambda_0), N_-\rangle)$$

and our equation has a solution, then

$$d\langle u_1(\lambda_0), N(x)\rangle < 0$$

for sufficiently large $||x||$.

4 Application to BVPs for ordinary differential equations

The most typical resonant BVP for ordinary differential equations is:

$$x'' + m^2 x = f(t, x, x'), \qquad x(0) = x(\pi) = 0, \tag{4.1}$$

where $f : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ and $m \in \mathbb{N}$. Assume that f is a Carathéodory function and satisfies the growth condition:

$$|f(t, x, y)| \leq a_M(t), \qquad |x|, |y| \leq M, \quad t \in [0, \pi],$$

where $a_M \in L^2(0,\pi)$. The problem is resonant, since for $f=0$ we have a nontrivial solution $t \mapsto \sin mt$ and the multiplicity of the resonance is $n=1$. Set $w_1(t) = \sin mt$. Let us assume that there exist limits

$$f_\pm(t) := \lim_{r\to\pm\infty} f(t, r\sin mt, mr\cos mt),$$

(it implies the sublinearity of the nonlinear term) and the numbers

$$\begin{array}{c} \int_{\sin mt>0} f_+(t)\sin mt\,dt + \int_{\sin mt<0} f_-(t)\sin mt\,dt, \\ \int_{\sin mt>0} f_-(t)\sin mt\,dt + \int_{\sin mt<0} f_+(t)\sin mt\,dt \end{array} \tag{4.2}$$

have the opposite signs. Put $X = H^1(0,\pi)$, $Y = H^2 \cap H_0^1$, $Z = L^2(0,\pi)$, $Lx = x'' + m^2 x$, $N(x) = f(\cdot, x(\cdot), x'(\cdot))$,

$$\langle u_1, z\rangle = \int_0^\pi z(t)\sin mt\,dt.$$

By Theorem 3.3 a solution to 4.1 exists. We perturb operator L by λI or $-\lambda I$ in order to get the appropriate sign of $c(\lambda)$. Limits $f_\pm$ can be finite or not but none of the numbers 4.2 can be a form $\infty - \infty$.

If f is sublinear, i.e.

$$\lim_{M\to\infty} \frac{||a_M||}{M} = 0,$$

and does not depend on the derivative x', one can put

$$f_+(t) = \liminf_{r\to+\infty} f(t, r\sin mt), \qquad f^-(t) = \liminf_{r\to-\infty} f(t, r\sin mt). \tag{4.3}$$

The first number in 4.2 should be positive and the second (with f^- instead of f_-) – negative, or reversely, if we exchange the upper and lower limits.

The problem is more difficult when f has a linear growth, i.e.

$$\lim_{M\to\infty} \frac{||a_M||}{M} = \gamma \in (0,\infty).$$

Assume that there exist positive constants $\check{a} < \hat{a}$, M and function $b \in L^2$ such that

$$\check{a} \le (f(x,u) - b(x))/u \le \hat{a} \tag{4.4}$$

for $|u| \ge M$. L is a self-adjoint operator in the Hilbert space $X = Z = L^2(0,\pi)$ and from the Hilbert-Schmidt theory, we are able to get a simple formula for the inverse operator

$$G_0(\lambda)z = \sum_{\lambda_s \in \mathrm{Sp}\,L} (\lambda_s - \lambda)^{-1}(w_s, z)w_s,$$

where w_s are eigenfunctions corresponding to eigenvalues λ_s. The formula for its norm is

$$||G_0(\lambda)|| = \max_{\lambda_s \in \mathrm{Sp}\,L} |\lambda_s - \lambda|^{-1}.$$

This method can be applied to more general BVP but here we have:

$$\lambda_s = -s^2, \quad w_s = \sqrt{\frac{2}{\pi}}\sin st, \quad s \in \mathbb{N}, \quad ||G_0(\lambda_0)|| = \frac{1}{2m-1}.$$

The formula for the norm is valid for $m > 1$; if $m = 1$ then this norm equals $1/3$.

Theorem 4.1. *Under the above assumptions, if for $m > 1$*

$$\gamma < (2m-1)\frac{\check{a}}{\sqrt{\check{a}^2 + (\hat{a} - \check{a})^2}},$$

and the Landesman-Lazer type condition 4.2 is satisfied, then problem 4.1 has a solution. For $m = 1$ in the last inequality one should replace $2m - 1$ by 3.

The proof is obtained by applying Theorem 3.2. Unfortunately, in the L.-L. condition for $m > 1$ we have the forbidden case $\infty - \infty$ and this theorem is valuable for $m = 1$ only.

One can use the space of continuous functions with the sup-norm instead of L^2. It is possible, obviously, under assumption of the continuity of f. We cannot find the exact formula for the norm of the inverse but the estimate only:

$$||G_0(\lambda_0)|| \leq \frac{\pi}{3m}(1 + 2\sqrt{2}) < \frac{4.02}{m}.$$

In order to get this estimate, we use as above the Hilbert-Schmidt theory

$$G_0(\lambda_0 = m^2)z = \sum_{j \neq m} (m^2 - j^2)^{-1}(e_j, z)e_j,$$

where $e_j(t) = \sqrt{2/\pi}\sin jt$, $j = 1, 2, \ldots$, and $(\cdot,\cdot)$ stand for the L^2-scalar product. Then, we find a bound for its supremum. We use notation 4.3 but with the assumption that both limits are uniform w.r.t. t. Take a positive number $\sigma < 1$ such that

$$4.02\gamma < m\frac{\sigma}{1+\sigma}.$$

Let us divide the interval $(0, \pi)$ into three pieces:

$$A^0_\sigma := \{t : |\sin mt| \leq \sigma\}, \quad A^+_\sigma := \{t : \sin mt > \sigma\}, \quad A^-_\sigma := \{t : \sin mt < -\sigma\}.$$

Denote

$$M_\sigma := \int_{A^0_\sigma} |\sin mt| \sup_{x \in \mathbb{R}} |f(t,x)|\, dt < \infty.$$

The L.-L. condition has the following form

$$\int_{A^+_\sigma} f_+(t)\sin mt\, dt + \int_{A^-_\sigma} f^-(t)\sin mt\, dt > M_\sigma,$$

$$\int_{A^+_\sigma} f^-(t)\sin mt\, dt + \int_{A^-_\sigma} f_+(t)\sin mt\, dt < -M_\sigma.$$

All the above results can be generalize to the vector equations

$$x_i'' + m_i^2 x_i = f_i(t, x, x'), \qquad x_i(0) = x_i(\pi) = 0, \quad i \leq k$$

provided that at least one of m_i is an integer to have a resonance.

Another kind of BVP is the periodic problem for the first order ODE in $\mathbb{R}^n$:

$$x' = f(t,x), \qquad x(0) = x(T), \tag{4.5}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0,\cdot) = f(T,\cdot)$ and the Carathéodory conditions. Usually one does not look at it as a resonant BVP but it is such, since the kernel of the linear part is the subspace of constant functions with values in $\mathbb{R}^n$. Let us assume that f satisfies the following growth condition

$$||f(t,x)|| \le a||x||^\rho + b(t), \qquad t \in [0,T], \quad x \in \mathbb{R}^n, \tag{4.6}$$

where $a > 0$, $\rho \in [0,1]$, $b \in L^1(0,T)$. Put $X = Z = L^1(0,T;\mathbb{R}^n)$, $Y = \{x : [0,T] \to \mathbb{R}^n$, x is absolutely continuous, $x(0) = x(T)\}$, $L(\lambda)x = x' - \lambda x$, $\lambda_0 = 0$, $N : X \to Z$ – the superposition operator. Then, after easy calculations,

$$\begin{aligned} &G_0(\lambda)z(t) = \mathrm{e}^{\lambda t}\textstyle\int_0^t \mathrm{e}^{-\lambda s} z(s)\,ds \\ &+(1-\mathrm{e}^{\lambda T})^{-1}\mathrm{e}^{\lambda(t+T)}\textstyle\int_0^T \mathrm{e}^{-\lambda s} z(s)\,ds + (T\lambda)^{-1}\int_0^T z(s)\,ds, \end{aligned}$$

$$\langle u_j(\lambda), z\rangle = T^{-1}\int_0^T z_j(s)\,ds, \qquad c_j(\lambda) = -\lambda^{-1},$$
$$w_j(\lambda)(t) = (\delta_{ij})_{j\le n},$$

$j = 1,\dots,n$. If $\rho < 1$, then we have the sublinear case. Condition 3.7 has the form: for any sequence $(x_k) \subset L^1(0,T;\mathbb{R}^n)$, $||x_k|| \to \infty$, $||x_k||^{-1}x_k \to d \in \mathbb{R}^n$, there exists j such that $d_j \ne 0$ and for large $k \in \mathbb{N}$:

$$d_j \int_0^T f_j(t, x_k(t))\,dt \ge 0. \tag{4.7}$$

One can assume, instead of 4.7:

$$\int_0^T x_{k,j}(t) f_j(t, x_k(t))\,dt > 0.$$

One can also add the left-hand side and obtain the following L.-L. condition with the scalar product:

$$\int_0^T (x_k(t), f(t, x_k(t)))dt > 0$$

for large k. In [67] we have shown, that the Mawhin's method leads to the condition:

$$(d, f(t,d)) > 0, \qquad \text{for } ||d|| = R, \quad t \in [0,T],$$

where R is a positive constant. Both conditions are incomparable.

Now, we consider the case of linear growth ($\rho = 1$ in 4.6). We need an estimate for $||G_0(0)||$. Since all calculations are easier in the supremum norm, we pass to the space of continuous functions, what is possible if f is continuous and

$$\gamma := \limsup_{||x||\to\infty} \sup_t \frac{||f(t,x)||}{||x||} \in (0,\infty).$$

In $\mathbb{R}^n$ we use the norm $||x|| = \max_{1\le i\le n} |x_i|$. We have

$$G_0(0)z(t) = \frac{1}{2}\int_0^t z(s)\,ds - \frac{1}{2}\int_t^T z(s)\,ds - \frac{1}{T}\int_0^T (t-s)z(s)\,ds,$$

hence $||G_0(0)|| \le T$. In order to apply Theorem 3.4, we need the sign of

$$\int_0^T f_j(t, \tilde{x}(t) + d)\,dt$$

for $||\tilde{x}(t)|| \le \sigma||d||$, $t \in [0,T]$, when the greatest coordinate of d is d_j, i.e. $||d|| = |d_j|$ and this number is sufficiently large. If $\sigma > 1$, then the sign of $\tilde{x}_j(t)+d_j$ is determined by the sign of d_j. Moreover, then $|\tilde{x}_i(t) + d_i| \le 2|d_j|$, $i \ne j$. Therefore the following condition is sufficient for the solvability of 4.5: for $j \in \{1, \dots, n\}$,

$$\begin{array}{ll} \liminf_{x_j\to+\infty} \inf_{|x_i|\le 2|x_j|} \int_0^T f_j(t, x_1, \dots, x_n)dt > 0 \\ > \quad \limsup_{x_j\to-\infty} \sup_{|x_i|\le 2|x_j|} \int_0^T f_j(t, x_1, \dots, x_n)dt, \end{array}$$

and (to get the estimate $\gamma||G_0(0)|| < \sigma/(1+\sigma) < \frac{1}{2}$)

$$2\gamma T < 1.$$

All inequalities in the L.-L. conditions can be reversed. In the very similar way one can study the periodic problem for the second order ODE:

$$x'' = f(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T).$$

One can observe more interesting phenomenon for

$$x'' + m^2 x = f(t, x), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \tag{4.8}$$

where $m \in \mathbb{N}$, even in 1-dimensional space. For the last problem, the kernel of the linear part is 2-dimensional with the basis

$$w_1(t) = \sin mt, \qquad w_2(t) = \cos mt.$$

Both these functions „interferes" one with the other. One can apply Theorem 3.2 assuming

$$|f(t,x)| \le a_M(t), \qquad |x| \le M, \quad t \in [0, 2\pi],$$

where $a_M \in L^1(0, 2\pi)$ and $\lim_{M\to\infty} M^{-1}||a_M||_{L^1} = 0$. Problem 4.8 has a solution, if there exist limits $f_\pm(t) = \lim_{x\to\pm\infty} f(t,x)$ and, for $(d_1, d_2) \in \mathbb{R}^2$, $d_1^2 + d_2^2 = 1$, we have: the numbers

$$\int_{A_+} wf_+ + \int_{A_-} wf_-, \qquad \int_{A_+} wf_- + \int_{A_-} wf_+$$

have the opposite signs, where $w(t) = d_1 \sin mt + d_2 \cos mt$, $A_\pm = \{t : \mathrm{sgn}(d_1 \sin mt + d_2 \cos mt) = \pm 1\}$.

Example 1. Set $f(t,x) = p(t)g_1(x^+) + r(t)g_2(x^-)$, where p and r belong to $L^\infty(0,2\pi)$, $g_1, g_2 : [0,\infty) \to \mathbb{R}$ are continuous and

$$g_1(0) = g_2(0),$$

$$\lim_{x\to\infty} g_j(x)/x = 0, \quad \lim_{x\to\infty} g_j(x) = \gamma_j \in (0,\infty\rangle, \qquad j = 1,2.$$

Then, if $p(t) \le 0 \le r(t)$ almost everywhere and at least one of two inequalities $p(t) < 0$, $0 < r(t)$ hold on the set of positive measure, then problem 4.8 has a solution. Similarly, for $p(t) = r(t) = 0$, $t \in (\pi, 2\pi]$, $\gamma_1 = \gamma_2 = \infty$ and

$$\int_0^\pi p(t) \sin t \, dt < 0.$$

If the nonlinear term f in 4.8 has a linear growth, we have no means to get L.-L. condition. Here, one can better see an interference of w_1 and w_2.

5 Problems of another kind

One can apply the perturbation method to the general boundary value problem for ODE. We follow [68]. Let E be a Banach space, $A : [0,1] \to L(E)$ – a continuous function taking values in the space of bounded linear operators of E, $f : [0,1] \times E \to E$ – a continuous function, $B_1,\ B_2 \in L(E)$, and let $B_3 : C([0,1],E) \to E$ be a nonlinear continuous mapping defined on the Banach space of continuous functions $[0,1] \to E$. We look for a solution of the first order differential equation

$$x' - A(t)x = f(t,x) \tag{5.1}$$

satisfying the boundary condition

$$B_1 x(0) + B_2 x(1) = B_3(x). \tag{5.2}$$

System 5.1-5.2 is at resonance, which means that the linear homogeneous problem

$$x' - A(t)x = 0, \quad B_1 x(0) + B_2 x(1) = 0,$$

has a nonzero solution. We shall assume that there exists an operator $A_0 \in L(E)$ commuting with the resolvent $U : [0,1] \to L(E)$ of the operator $x' - A(t)x$ such that $B_1 + B_2 \exp \lambda A_0 U(1)$ is an authomorphism of E for λ from a neighbourhood (nhbd) of $0 \in \mathbb{R}$. Usually, $A_0 = I$ – the identity operator. Moreover, let $B_1 + B_2 U(1)$ be a linear Fredholm operator (its index must be 0 by the above). Our assumptions mean that the problems

$$x' - A(t)x - \lambda A_0 x = 0, \quad B_1 x(0) + B_2 x(1) = 0, \tag{5.3}$$

have only the zero-solution for $\lambda \neq 0$ belonging to the nhbd of 0, the subspace of initial points of solutions to 5.3 with $\lambda = 0$ is finite dimensional, and that the range of the operator $B_1 + B_2 U(1)$ has a finite codimension.

Take any basis $x_1, \dots, x_n$ in $\ker(B_1 + B_2 U(1))$ and suppose that the following limits

$$\lim_{\lambda \to 0} B(\lambda) x_j / ||B(\lambda) x_j|| =: h_j, \quad j = 1, \dots, n, \tag{5.4}$$

where $B(\lambda) = B_1 + B_2 U(1) \exp \lambda A_0$, exist and constitute a linearly independent system such that

$$\mathrm{Lin}\{h_1, \dots, h_n\} \oplus \mathrm{im} B(0) = E.$$

Then, of course, this condition is satisfied for each basis.

Let $E_1 = \ker B(0)$ and let E_0 be its topological complement:

$$E_1 \oplus E_0 = E.$$

We have $B(\lambda)E_1 \oplus B(\lambda)E_0 = E$ for $\lambda \neq 0$ sufficiently close to 0. Moreover,

$$\mathrm{Lin}\{h_1, \dots, h_n\} \oplus B(0)E_0 = E.$$

Define the system of linear bounded functionals on $E : v_j(\lambda)$, $j = 1, \dots, n$, for $\lambda \neq 0$ by the formulas

$$\langle v_j(\lambda), B(\lambda) x_i \rangle = \delta_{ij} ||B(\lambda) x_i||, \quad i = 1, \dots, n,$$

$$v_j(\lambda) \mid B(\lambda) E_0 = 0.$$

Obviously, v_j are continuous functions of λ and have continuous extensions to 0 such that

$$\langle v_j(0), h_i \rangle = \delta_{ij}, \quad v_j(0) \mid B(0) E_0 = 0.$$

If we denote by $P_1(\lambda)$ (resp. $P_0(\lambda)$) the projectors on $B(\lambda)E_1$ (resp. $B(\lambda)E_0$) along $B(\lambda)E_0$ (resp. $B(\lambda)E_1$) for $\lambda \neq 0$ and similarly for $\lambda = 0$ with natural changes, then we can find the representation of $B(\lambda)^{-1}$:

$$B(\lambda)^{-1} = B(\lambda)^{-1} P_0(\lambda) + \sum_{j=1}^{n} ||B(\lambda) x_j||^{-1} \langle v_j(\lambda), \cdot \rangle x_j$$

where the first summand has a continuous extension to $0 : (B(0) \mid E_0)^{-1} P_0(0)$. We shall denote this summand by $R(\lambda)$, and

$$c_j(\lambda) := ||B(\lambda) x_j||^{-1}, \quad j = 1, \dots, n,$$

are the only parts which make $\lambda = 0$ a singular point of $B(\lambda)^{-1}$. We have

$$B(\lambda)^{-1} = R(\lambda) + \sum_{j=1}^{n} c_j(\lambda) \langle v_j(\lambda), \cdot \rangle x_j \tag{5.5}$$

which is similar to the corresponding formula from the previous sections.

It is easy to see that $V_\lambda(t) = \exp(\lambda t A_0)U(t)$ is the resolvent for the operator $x' - A(t)x - \lambda A_0 x$. This implies that the unique solution to the BVP

$$x' - A(t)x - \lambda A_0 x = b(t), \quad B_1 x(0) + B_2 x(1) = 0,$$

is the function

$$x(t) = V_\lambda(t)x_0 + V_\lambda(t)\int_0^t V_\lambda^{-1}(s)b(s)\,ds$$

with the initial vector x_0 for which

$$B(\lambda)x_0 \;=\; -B_2\,\exp(\lambda A_0)U(1)\int_0^1 \exp(-\lambda s A_0)U^{-1}(s)b(s)\,ds.$$

We shall denote the right-hand side of the last equality by $C(\lambda, b)$ where $b \in C([0,1], E)$. Applying 5.5, we get, for $\lambda \neq 0$,

$$x_0 = R(\lambda)C(\lambda, b) + \sum_{j=1}^{n} c_j(\lambda)\langle v_j(\lambda), C(\lambda, b)\rangle x_j$$

and

$$\begin{array}{rrcl} & x(t) & = & \exp(\lambda t A_0)U(t)R(\lambda)C(\lambda, b) + \\ (5.6) & & + & \exp(\lambda t A_0)U(t)\int_0^t \exp(-\lambda s A_0)U^{-1}(s)b(s)\,ds + \\ & & + & \sum_{j=1}^{n} c_j(\lambda)\langle v_j(\lambda), C(\lambda, b)\rangle \exp(\lambda t A_0)U(t)x_j. \end{array}$$

Now, we are able to write down the system equivalent to the BVP

$$x' - A(t)x - \lambda A_0 x = f(t, x), \quad B_1 x(0) + B_2 x(1) = B_3 x,$$

for $\lambda \neq 0$:

$$\begin{array}{rrcl} & x_0 & = & R(\lambda)(C(\lambda, N(x)) + B_3(x)) + \\ (5.7) & & + & \displaystyle\sum_{j=1}^{n} c_j(\lambda)\langle v_j(\lambda), C(\lambda, N(x)) + B_3(x)\rangle x_j, \end{array}$$

$$\begin{array}{rrcl} & x(t) & = & V_\lambda(t)R(\lambda)(C(\lambda, N(x)) + B_3(x)) + \\ & & + & V_\lambda(t)\displaystyle\int_0^t V_\lambda^{-1}(s)N(x)(s)\,ds + \\ (5.8) & & + & \displaystyle\sum_{j=1}^{n} c_j(\lambda)\langle v_j(\lambda), C(\lambda, N(x)) + B_3(x)\rangle V_\lambda(t)x_j, \end{array}$$

where $N(x)(t) = f(t, x(t))$. We can show that the operator defined by the right-hand sides of 5.7, 5.8 on $E \times C([0,1], E)$ is completely continuous, if $f(t, \cdot)$ is completely continuous, functions $f_x := f(\cdot, x)$ are equicontinuous for x belonging to each bounded set and B_3 is completely continuous. Then, we can find solutions to

5.7-5.8 for $\lambda \neq 0$ if f and B_3 are sublinear, by the Rothe Fixed Point Theorem [19], and prove that the existence of a bounded sequence of solutions for $\lambda_m \to 0$ implies the solvability of the studied resonant problem. Next, we should find conditions excluding the existence of unbounded sequence of solutions (the Landesman-Lazer type condition). Here it is:

for any $(x^m) \subset C([0,1], E)$ with the properties $||x^m||_\infty \to \infty$ (the supremum norm in $C([0,1], E)$), $||x^m||_\infty^{-1} x^m \to \sum d_j U(\cdot) x_j$ for some $(d_1, \dots, d_n) \in \mathbb{R}^n$, there exists $j \in \{1, \dots, n\}$ such that

$$\limsup_{m\to\infty} d_j \langle v_j(0), D(x^m)\rangle \ < 0$$

where

$$D(x) = -B_2 U(1) \int_0^1 U^{-1}(s) f(s, x(s))\, ds + B_3(x).$$

The proof is similar to that of Theorem 3.2. The case of nonlinearities with a linear growth is examined separately by homotopy arguments in the above mentioned [68].

BVPs of different kinds are considered with additional property of a solution: the positivity. In our abstract framework, it means that one look for a solution belonging to a prescribed cone, i.e. a closed set $P \subset X$ (the space of functions) with properties:

$$x_1, x_2 \in P \Longrightarrow x_1 + x_2 \in P, \qquad x \in P,\ \lambda \geq 0 \Longrightarrow \lambda x \in P, \qquad P \cap (-P) = \{0\}.$$

In applications, we usually think about $P \subset C([a,b], \mathbb{R}^n)$ or $P \subset L^2([a,b], \mathbb{R}^n)$ with all coordinates $x_j(t) \geq 0$. Our method work also for problems of this kind [66]. We need some assumptions on the behaviour of the nonlinear term on the boundary of the cone and define another nonlinearity which equals the given one on the cone. A solution for the second equation exists by our method and, a possibility that it sticks out the cone is excluded. Thus, it is a solution of the given problem.

Similar applications, as presented in the previous section, of Theorems 3.2 and 3.4 can be obtained for elliptic partial differential equations. One have to work in appropriate Sobolev or Hölder spaces but all calculations and arguments are more complicated.

6 Functional-differential equations

Equations of this kind are of the great interest recently, and the class of such equations is very large. It includes equations with delay, integro-differential equations and many others. Relatively general formulation of boundary value problems for functional-differential equations can be found in [53]. Our method can be applied to this kind of problems.

Let $C = C([-r, 0], \mathbb{R}^k)$ and for $t \in [0,1]$ and $x : [-r, 1] \to \mathbb{R}^k$ put $x_t(s) = x(t+s)$, $s \in [-r, 0]$. Let $f : [0,1] \times C \to \mathbb{R}^k$ satisfies the following Carathéodory conditions:

- $f(\cdot, \varphi)$ is measurable for $\varphi \in C$,

- $f(t,\cdot)$ is continuous on C, and the following growth condition
- for any $M>0$ there exists $a_M \in L^1(0,1)$ such that

$$||f(t,\varphi)|| \le a_M(t), \qquad \text{for } \ t\in[0,1], \ \ ||\varphi|| \le M. \tag{6.1}$$

Take $A, B \in L(C)$ and $\Phi : C \times C \to C$ being nonlinear completely continuous operator. We are interested in the solvability of the following BVP:

$$x' = f(t, x_t), \tag{6.2}$$

$$Ax_0 + Bx_1 = \Phi(x_0, x_1). \tag{6.3}$$

By a solution we mean a function $x : [-r,1] \to \mathbb{R}^k$ continuous and such that $x|[0,1]$ is absolutely continuous, for almost all t, one has $x'(t) = f(t, x_t)$, and the boundary condition 6.3 is satisfied ($x_0 = x|[-r,0]$, $x_1(s) = x(s+1)$ for $s \in [-r,0]$). Now, we shall perturb the boundary condition but not the f.-d. equation. Suppose we have a continuous family of linear bounded operators $A(\lambda)$, λ in a nhbd of 0, $A(0) = A$. Put $X = C([-r,1],\mathbb{R}^k)$, $Y = \{x \in X : x|[0,1] \in W^{1,1}(0,1;\mathbb{R}^k)\}$, $Z = L^1(0,1;\mathbb{R}^k) \times C$, $L(\lambda)x = (x'|[0,1],\ A(\lambda)x_0 + Bx_1)$, $N(x) = (f(\cdot, x_\cdot),\ \Phi(x_0,x_1))$. Introduce two operators $S \in L(C)$, $T \in L(Z,C)$ by formulas

$$S\varphi(s) = \begin{cases} \varphi(s+1) & \text{for } s \le -1, \\ 0 & \text{for } s \in (-1,0], \end{cases}$$

$$Tz(s) = \begin{cases} 0 & \text{for } s \le -1, \\ \int_0^{s+1} z(t)dt & \text{for } s \in (-1,0]. \end{cases}$$

Since the following initial problem: $x' = z(t)$, $x|[-r,0] = \varphi \in C$ has a solution

$$x(s) = \begin{cases} \varphi(s), & s \in [-r,0], \\ \varphi(0) + \int_0^s z(t)dt, & s \in (0,1], \end{cases}$$

and the inverse operators to $L(\lambda)$ can be computed if we find a solution to $x' = z(t)$, $A(\lambda)x_0 + Bx_1 = \psi$, then the initial function φ should satisfy the equation $[A(\lambda)+BS]\varphi = \psi - BTz$. Thus, if the operators $A(\lambda)+BS$ are invertible for $\lambda \ne 0$ and $A+BS$ is Fredholm operator (it has index 0), then also $L(\lambda)$ hold the same conditions. Take a basis of $\ker(A+BS) \subset C : \varphi_1, \dots, \varphi_n$. In order to obtain the needed form of the inverse operators, assume there exist the limits

$$\lim_{\lambda\to 0} ||[A(\lambda)+BS]\varphi_j||^{-1}[A(\lambda)+BS]\varphi_j =: h_j, \qquad j = 1,\dots,n,$$

and

$$\mathrm{Lin}\{h_1,\dots,h_n\} \oplus \mathrm{im}\,(A+BS) = C. \tag{6.4}$$

Then

$$[A(\lambda) + BS]^{-1} = C_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle v_j(\lambda), \cdot\rangle\varphi_j,$$

with $C_0(\lambda) \in L(C)$, $v_j(\lambda) \in C^*$ for $j = 1, \dots, n$, having continuous extensions to $\lambda = 0$, and $c_j(\lambda) \to +\infty$ as $\lambda \to 0$. We can put in our scheme:

$$G_0(\lambda)(z, \psi)(s) = \begin{cases} C_0(\lambda)(\psi - BTz)(s) & \text{for } s \in [-r, 0], \\ C_0(\lambda)(\psi - BTz)(0) + \int_0^s z(t)dt & \text{for } s \in (0, 1], \end{cases}$$

$$\langle u_j(\lambda), (z, \psi)\rangle = \langle v_j(\lambda), \psi - BTz\rangle,$$

$$w_j(s) = \begin{cases} \varphi_j(s) & \text{for } s \in [-r, 0], \\ \varphi_j(0) & \text{for } s \in (0, 1], \end{cases}$$

where $j = 1, \dots, n$. Since $v_j(0)$ are linear bounded functionals on the space of continuous functions, there exist functions with bounded variation g_j, $j = 1, \dots, n$, with the property

$$\langle v_j(0), \varphi\rangle = \int_{-r}^{0} \varphi dg_j \qquad \text{the Lebesgue-Stieltjes integral.}$$

Denote the nonlinear functional

$$\Psi(\varphi)(s) = \begin{cases} 0 & \text{for } \ s \le -1, \\ \int_0^{s+1} f(t, \varphi_t)dt & \text{for } \ s \in (-1, 0]. \end{cases}$$

If the nonlinearity f is sublinear, i.e.

$$\lim_{M\to\infty} M^{-1}||a_M||_{L^1} = 0,$$

$$\lim_{\varphi,\psi\in C,\ ||\varphi||\to\infty,\ ||\psi||\to\infty} \frac{||\Phi(\varphi, \psi)||}{(||\varphi|| + ||\psi||)} = 0,$$

then the L.-L. condition guaranteeing the solvability of the problem 6.2-6.3 has the form:

for any sequence $(x_\nu) \subset X$ with the properties $||x_\nu|| \to \infty$, $||x_\nu||^{-1}x_\nu \to \sum d_i w_i$, there exists $j \in \{1, \dots, n\}$ such that $d_j \neq 0$ and

$$d_j \int_{-r}^{0} [\Phi((x_\nu)_0, (x_\nu)_1) - B\Psi((x_\nu)_0)]dg_j \le 0$$

for sufficiently large $\nu \in \mathbb{N}$.

If there exist limits $\lim_{\nu\to\infty} \Phi((x_\nu)_0, (x_\nu)_1) =: \Phi_d$ and $\lim_{\nu\to\infty} f(t, (x_\nu)_t) =: f_d(t)$ depending only on $d \in \mathbb{R}^n$, $||d|| = 1$, then our problem has a solution, if for any $d \in \mathbb{R}^n$, $||d|| = 1$, there exists j such that

$$d_j \int_{-r}^{0} [\Phi_d - B\Psi(d)]dg_j < 0,$$

where $\Psi(d)$ stands for a function in C that equals 0 for $s \le -1$ and

$$\Psi(d)(s) = \int_0^{s+1} f_d(t)dt, \qquad s \in (-1, 0].$$

One can write down an appropriate corollary of Theorem 3.4 for the nonlinearities with the linear growth.

Let us consider a particular case of the problem 6.2-6.3 – the periodic problem

$$x' = f(t, x_t), \qquad x_0 = x_1. \tag{6.5}$$

If f is 1-periodic with respect to t, i.e. $f(t, \varphi) = f(t+1, \varphi)$ for $\varphi \in C$, then any solution of 6.5 has a 1-periodic extension on $\mathbb{R}$. The remaining assumptions on f are withour changes.

Put $A(\lambda)\varphi = (1+\lambda)\varphi$, $B\varphi = -\varphi$, $\Phi = 0$. It is easy to check that $A(\lambda) + BS$ is invertible for $\lambda \neq 0$, $A(0) + BS$ has the kernel built of constant functions and the range with functions such that $\psi(0) = 0$. One can take

$$\varphi_j(t) = (\delta_{ij})_{i \le k}.$$

For $\lambda \to 0^+$, we obtain $h_j = \varphi_j$, $j = 1, \dots, n$, and condition 6.4 holds. Functionals $v_j(0)$ are given by functions $g_j = (0, \dots, 0, \Theta, 0, \dots, 0)$, where Θ is the Heaviside's function and the integral over $[-r, 0]$ reduces to the value of the integrand at $s = 0$. At last the L.-L. condition for the existence of a solution to 6.5 has a form:

for any sequence $(x_\nu) \subset C([0,1], \mathbb{R}^k)$ such that $||x_\nu|| \to \infty$, $||x_\nu||^{-1}x_\nu \to d \in \mathbb{R}^k$ uniformly, there exists $j \in \{1, \dots, n\}$ such that $d_j \neq 0$ and

$$d_j \int_0^1 f_j(t, (x_\nu)_t)dt \le 0$$

for sufficiently large ν. Obviously, the weaker condition

$$\limsup_{\nu \to \infty} \int_0^1 (x_\nu(t), f(t, (x_\nu)_t))dt < 0$$

or, in the case of the existence of limits $\lim_{\nu \to \infty} f(t, (x_\nu)_t) = f_d(t)$ depending on $d = \lim ||x_\nu||^{-1}x_\nu \in \mathbb{R}^k$ only:

$$\int_0^t (d, f_d(t))dt < 0,$$

are suffiecient for the existence of a solution. If we take the left-hand side limits $\lambda \to 0^-$ instead of the right-hand ones, then $h_j = -\varphi_j$ and instead of g_j we get $-g_j$. As the result all inequalities in the L.-L. conditions should be reversed.

In the similar way, one can study the BVPs for differential equations with delay or neutral such as

$$x'' + m^2 x = f(t, x, x_h), \qquad x(0) = x(\pi) = 0,$$

or

$$x_h'' + m^2 x_h = f(t, x, x_h), \qquad x(0) = x(\pi) = 0,$$

and also impulsive differential equations.

7 Multiple solutions of nonlinear equations with parameter near bifurcation point

We shall deal with a continuous family of nonlinear equations

$$L(\lambda)x = N(x) \tag{7.1}$$

where $L(\lambda) : Y \to Z$ are linear bounded operators, $Y \subset X$ is a linear subspace and $N : X \to Z$ is nonlinear and continuous, $\lambda \in \operatorname{nhbd} \lambda_0$. All assumptions about this family of operators are almost the same as in section 3. Especially, it has an isolated singularity in λ_0, i.e. $L(\lambda)$ are invertible for $\lambda \neq \lambda_0$ and $L(\lambda_0)$ is a linear Fredholm operator of index zero. Moreover, we assume a special form of the family

$$L(\lambda)^{-1} = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j, \cdot\rangle w_j$$

where $G_0(\lambda) \in L(Z, X)$ has a continuous extension on λ_0, $c_j(\lambda) \in \mathbb{R}$ and

$$\lim_{\lambda \to \lambda_0} |c_j(\lambda)| = \infty,$$

the vectors $w_1, ..., w_n$ forms a basis of the kernel $\ker L(\lambda_0)$, $u_1, \dots, u_n$ are linearly independent covectors on Z anihilating the range $\operatorname{im} L(\lambda_0)$. The above abstract scheme can be applied to a great variety of boundary value problems such as, in particular, those considered by Mawhin and Schmitt [56]:

$$x'' + \lambda x + g(t, x) = h(t), \;\; x(0) = x(2\pi), \; x'(0) = x'(2\pi),$$

where $\lambda_0 = 0$,

$$x'' + \lambda x + g(t, x) = h(t), \;\; x(0) = x(\pi) = 0, \;\; \text{where } \lambda_0 = 1.$$

It is well known that these problems have at least one solution for $\lambda = \lambda_0$ if g is integrally bounded and the Landesman-Lazer condition holds. In our abstract framework, this means that N is bounded (or sublinear) and, for any $(x_k) \subset X$ such that $||x_k|| \to \infty$, $||x_k||^{-1}x_k \to \sum_{i=1}^{n} d_i w_i$, there exists j with the property

$$d_j \operatorname{sgn}_{\lambda > \lambda_0} c_j(\lambda)\langle u_j, N(x_k)\rangle \leq 0$$

for large k (comp. section 3). This solution is an element of a continuum (in $X \times \mathbb{R}$) of solutions to 7.1 for $\lambda \geq \lambda_0$ and can be extended to λ less but close to λ_0. Let us assume that $G_0(\lambda)$ take values in topological complements of $\ker L(\lambda_0)$ and that the function

$$(-1)^p \det\langle u_i, L(\lambda)w_j\rangle$$

changes the sign when λ passes through λ_0, where p is the number of coefficients $c_j(\lambda)$, $j \leq n$, which has opposite signs on the left and on the right of λ_0. Then there exists a connected branch of solutions to 7.1 bifurcating from the infinity. It follows that there exists $\delta > 0$ such that 7.1 has at least two solutions for $\lambda \in (\lambda_0 - \delta, \lambda_0)$. All assumptions on $L(\lambda)$ are satisfied if there exist the limits

$$\lim_{\lambda \to \lambda_0^{\pm}} ||L(\lambda)w_j||^{-1} L(\lambda)w_j = \pm h_j$$

and $h_1, \ldots, h_n$ forms a basis of a topological complement of $L(\lambda_0)$ and n (the geometric multiplicity of the eigenvalue λ_0) is odd.

Mawhin and Schmitt [56] used conditions closed but different from the Landesman-Lazer one for their specific equations. They got at least three solutions, since they can distinguish positive and negative solutions with large norms. However they needed $n = 1$ and that all eigenfunctions have constant signs.

8 Multiple solutions – general case

The question of the existence of more than one solution to a nonlinear equation is usually more difficult than the existence problem of at least one solution. Starting from the well-known paper by Ambrosetti and Prodi [4] (see also [3]), there appeared a lot of works based on a similar picture: if a nonlinear map $N : X \to Z$ "folds the Banach space X in two and lays it on the Banach space Z", then the equation $N(x) = z$ has one solution for the z's corresponding the folding place, two solutions for the z's lying on one side of this place (being a 1-codimensional manifold) and no solution for the z's lying on the other side. Generalizations can be done in many directions. Roughly speaking, two kinds of them are possible – the first if N "folds" many times, the second if N is homotopically equivalent to the above standard map. Here, we study the second case.

Each abstract result needs some applications. Most of authors even omit abstract theorems and give results on typical boundary value problems. Often, it is the problem

$$\Delta u + f(u) = g(x), \qquad u|\partial\Omega = 0,$$

as in [4], [7], or G depends on an additional parameter [42], [50], [82], or the Laplace operator Δ is replaced by a more general elliptic operator and the Dirichlet boundary condition – by another one [5]. The operator acts as we need if f crosses the first eigenvalue λ_1 of the linear problem

$$\Delta u + \lambda u = 0, \qquad u|\partial\Omega = 0.$$

This last condition can be described by the asymptotic behaviour of $f(\xi)/\xi$ as ξ tends to $\pm\infty$, or by the behaviour of the derivative f'. The study of the problem is also possible if f crosses more than one eigenvalue (see [76], [51]).

Our approach is different and based on the abstract scheme introduced in [63], [65]. It was very fruitful when we asked about the existence of at least one solution (comp. [63]-[68]). We isolate a linear part L_0 which is a Fredholm operator with 1-dimensional kernel and cokernel and use homotopy arguments to get at least two solutions for z sufficiently far from the image of L_0 in one direction and no solution for such z's in the other one. The result is largely applicable and works for standard boundary value problems under different assumptions. Another kind of this theorem with applications can be found in [69]. However, the present result seems to be more natural from the applicational point of view.

Let X and Z be real Banach spaces, $L_0 : X \supset Y \to Z$ a linear Fredholm operator such that

$$\ker L_0 = \mathrm{Lin}\{w\}, \qquad \mathrm{im} L_0 = \ker u,$$

where $w \in Y$, $u \in Z^*$. Fix $h_0 \in Z$ such that $\langle u, h_0 \rangle = 1$. Let $N : X \to Z$ be a nonlinear continuous map which transforms bounded sets into bounded ones and such that

$$\gamma := \limsup_{||x|| \to \infty} \frac{||N(x)||}{||x||} \tag{8.1}$$

satisfies

$$\gamma ||(L_0|\tilde{X})^{-1}P|| < 1 \tag{8.2}$$

where $\tilde{X}$ is a topological complement of $\ker L_0$ and $P : Z \to Z$ denotes the linear projection onto $\mathrm{im} L_0$ along $\mathrm{Lin}\{h_0\}$. Let

$$X = \tilde{X} \oplus \overline{X}$$

where $\overline{X} = \mathrm{Lin}\{Jw\}$. Suppose that N satisfies the following condition:

(C) there exists $\beta > \frac{\gamma||(L_0|\tilde{X})^{-1}P||}{1-\gamma||(L_0|\tilde{X})^{-1}P||}$ such that

$$\lim_{d \to \pm\infty} \sup_{||\tilde{x}|| \le \beta ||dw||} \langle u, N(\tilde{x} + dw) \rangle = -\infty$$

where $\tilde{x} \in \tilde{X}$, $d \in \mathbb{R}$.

We are interested in the equation

$$L_0 y = N(Jy) + h_1 + t h_0 \tag{8.3}$$

where $h_1 \in \mathrm{im} L_0$ (hence $\langle u, h_1 \rangle = 0$) and $t \in \mathbb{R}$.

Theorem 8.1. *Under the above notations and assumptions, there exists $t_0 > 0$ (depending on h_1) such that equation 8.3 has no solution for $t \le -t_0$ and at least two solutions for $t \ge t_0$.*

Proof. The first part of the assertion has an elementary proof. Consider the system

$$\begin{aligned} &\tilde{x} = (L_0|\tilde{X})^{-1} P(N(x) + h_1), \\ &\langle u, N(x) \rangle = -t, \end{aligned} \tag{8.4}$$

where $x = \tilde{x} + dw$, which is equivalent to 8.3. Take $\varepsilon > 0$ so small that

$$(\gamma + \varepsilon)||(L_0|\tilde{X})^{-1}P||/(1 - (\gamma + \varepsilon)||(L_0|\tilde{X})^{-1}P||) < \beta$$

and $R > 0$ so large that, for $||x|| \geq R$, we have $||N(x)|| \leq (\gamma + \varepsilon)||x||$. If x satisfies system 8.4 and $||x|| \geq R$, then

$$||\tilde{x}|| \leq (\gamma + \varepsilon)||(L_0|\tilde{X})^{-1}P||(||\tilde{x}|| + ||dw||) + ||(L_0|\tilde{X})^{-1}Ph_1||,$$

hence

$$||\tilde{x}|| \leq \frac{(\gamma + \varepsilon)||(L_0|\tilde{X})^{-1}P||\ ||dw|| + ||(L_0|\tilde{X})^{-1}Ph_1||}{1 - (\gamma + \varepsilon)||(L_0|\tilde{X})^{-1}P||}.$$

So, there exists $d_0 > 0$ such that $||\tilde{x}|| \leq \beta||dw||$ for $|d| \geq d_0$. From the second equation of 8.4, $t < 0$ and condition (C) we get that there exists $d_1 > 0$ such that $|d| \leq d_1$. Therefore, for x satisfying 8.4, $||x|| \leq (\beta + 1)d_1||w||$ or $||x|| < R$. Recalling that N maps bounded sets onto bounded ones, we obtain that equation 8.3 has no solution if

$$t < -\sup\{|\langle u, N(x)\rangle| : \ ||x|| \leq \max(R, (\beta + 1)d_1||w||)\}.$$

For the proof of the second part of the assertion, we need some introductory remarks. As in [65], consider the continuous family of linear operators $L(\lambda) : Y \to Z$, $\lambda \in \mathbb{R}$, given by the formula

$$L(\lambda)(\tilde{x} + dw) = L_0\tilde{x} + \lambda dh_0$$

where $\tilde{x} \in \tilde{X}$. Obviously, $L(\lambda)$ are linear homeomorphisms for $\lambda \neq 0$ and $L(0) = L_0$. Moreover,

$$L(\lambda)^{-1}z = (L_0|\tilde{X})^{-1}Pz + \lambda^{-1}\langle u, z\rangle w \tag{8.5}$$

for $\lambda \neq 0$ and $z \in Z$. Instead of equation 8.3, we study the family of equations

$$L(\lambda)x = N(x) + h_1 + th_0, \qquad \lambda \neq 0. \tag{8.6}$$

They are equivalent to the systems

$$\begin{aligned} \tilde{x} &= (L_0|\tilde{X})^{-1}P(N(x) + h_1), \\ d &= \lambda^{-1}(\langle u, N(x)\rangle + t), \end{aligned} \tag{8.7}$$

which we slightly modify by putting

$$\begin{aligned} \tilde{x} &= \tfrac{\lambda \pm \alpha}{\pm \alpha}(L_0|\tilde{X})^{-1}P(N(x) + h_1), \\ d &= \lambda^{-1}(\langle u, N(x)\rangle + t), \end{aligned} \tag{8.8}$$

where α is a small positive constant defined later.

First of all, notice that if 8.8 has a solution $x_k = \tilde{x}_k + d_k w$ for $\lambda = \lambda_k$, $\lambda_k \to 0$, and the sequence (x_k) is bounded, then it has a subsequence converging to a solution $x = \tilde{x} + dw$ of system 8.4, hence it is a solution to equation 8.3.

If we find two sequences of such x_k's, the first with $d_k \geq \eta > 0$ and the second with $d_k \leq -\eta$, then the subsequent solutions $x = \tilde{x} + dw$ will have $d \geq \eta$ and $d \leq -\eta$, respectively, hence they will be different. In the sequel, we shall find two open bounded sets V_+ and V_- in X (with $d \geq \eta$ and $d \leq -\eta$, respectively) such that there are no solutions of 8.8 on the boundaries $\partial V_\pm$. For $\lambda = +\alpha$ (or $-\alpha$), we have a 1-dimensional map given by the right-hand side of the second equation of 8.8 which has the Brouwer degree ± 1. This implies the existence of two families x_λ^+ and x_λ^- of solutions to 8.8 in V_+ and V_-, respectively. They lead to two different solutions of our equation 8.3. This is an idea of the proof.

Suppose that $x = \tilde{x} + dw$ satisfies system 8.8 and take $\varepsilon > 0$ and $R > 0$ as above. Then $||x|| < R$ or $||\tilde{x}|| \leq \beta ||dw||$ for $|d| \geq d_0$ found in the first part of the proof. Enlarging $d_0 > 0$, if necessary, we have that $||x|| \geq R$ for $|d| \geq d_0$ and any $\tilde{x}$. Fix $t > 0$ such that

$$|\langle u, N(x)\rangle| < \frac{1}{2}t \qquad \text{for} \qquad ||x|| \leq (\beta + 1)d_0||w||,$$

and then choose $d_1 > d_0$ such that

$$\langle u, N(x)\rangle < -2t \qquad \text{for} \qquad |d| \geq d_1, \quad ||\tilde{x}|| \leq \beta||dw||.$$

Define open bounded sets

$$V_+ = \{\tilde{x} + dw : \ d \in (d_0, d_1), \ \ ||\tilde{x}|| < \beta d_1 ||w|| + 1\},$$

$$V_- = \{\tilde{x} + dw : \ d \in (-d_1, -d_0), \ \ ||\tilde{x}|| < \beta d_1 ||w|| + 1\}.$$

Then $\partial V_\pm = \{x : \ d = \pm d_0 \ \text{ or } \ d = \pm d_1 \ \text{ or } \ ||\tilde{x}|| = \beta d_1 ||w|| + 1\}$. If x satisfies system 8.8 with lower signs $(-\alpha)$, then $x \notin \partial V_+$. In fact, $d = d_1$ is excluded by the comparison of the signs in the second equation of the system, $||\tilde{x}|| = \beta d_1 ||w|| + 1$ – by the inequality $||\tilde{x}|| \leq \beta ||dw|| \leq \beta d_1 ||w||$. The last possibility $d = d_0$ will be rejected if we take a sufficiently small $\alpha > 0$ so that

$$d_0 < \frac{1}{2}\alpha^{-1}t.$$

Therefore, the Leray-Schauder degree $\deg_{LS}(H(\cdot, \lambda), V_+, 0)$, where

$$H(x, \lambda) = x - \frac{\lambda - \alpha}{-\alpha}(L_0|\tilde{X})^{-1}P(N(x) + h_1) - \lambda^{-1}(\langle u, N(x)\rangle + t)w,$$

is independent of $\lambda \in (0, \alpha]$. For $\lambda = \alpha$, we have the 1-dimensional map and its Leray-Schauder degree is equal to the Brouwer degree

$$\deg(h_+, (d_0, d_1), 0)$$

where $h_+(d) = d - \alpha^{-1}(\langle u, N(dw)\rangle + t)$. Since $h_+(d_0) < 0$ and $h_+(d_1) > 0$, this last degree equals $+1$. Thus, for any $\lambda \in (0, \alpha]$, there exists $x_\lambda = \tilde{x}_\lambda + d_\lambda w$ such that $H(x_\lambda, \lambda) = 0$. This produces a solution $x = \tilde{x} + dw$ to system 8.4 with $d \geq d_0 > 0$.

Similar arguments for V_- and system 8.8 with upper signs give a solution to 8.4 with $d \leq -d_0$. We have obtained at least two solutions to our equation. $\square$

Remark 1. If $\gamma = 0$ (N is sublinear), condition (C) can be replaced by:

(C*) for any sequence $x_k = \tilde{x}_k + d_k w$, $k \in \mathbb{N}$, such that $||\tilde{x}_k||/d_k \to 0$ and $d_k \to \pm\infty$,

$$\lim_{k\to\infty} \langle u, N(x_k)\rangle = -\infty.$$

If we replace the value of the limits $-\infty$ by $+\infty$ in conditions (C) and (C*), then we shall get two solutions for large negative t's and no solution for large positive t's.

Let us notice that conditions (C) and (C*) are completely inconsistent with the Landesman-Lazer type conditions from section 3, where $\langle u, N(\tilde{x} + dw)\rangle$ changes the sign when d passes from large negative values to large positive ones.

We shall show some applications of the above result. First, we shall study the boundary value problem

$$x'' + m^2 x = f(t,x) + h_1(t) + s \sin mt, \qquad x(0) = x(\pi) = 0, \tag{8.9}$$

where $m = 1, 2, \ldots$, $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ and $h_1 : [0,\pi] \to \mathbb{R}$ are continuous functions with

$$\int_0^\pi h_1(t) \sin mt \, dt = 0$$

and $s \in \mathbb{R}$. Put $X = Z = C[0,\pi]$, $Y = \{x \in C^2[0,\pi] :\ x(0) = x(\pi) = 0\}$,

$$L_0 x = x'' + m^2 x, \qquad N(x)(t) = f(t, x(t)),$$

$h_0(t) = w(t) = \sin mt$,

$$\langle u, z\rangle = \int_0^\pi z(t) \sin mt \, dt.$$

It is easy to find the Green operator by using the Hilbert-Schmidt theory:

$$(L_0|\tilde{X})^{-1} P z = \sum_{j=1, j\neq m}^{\infty} (m^2 - j^2)^{-1} (e_j, z) e_j$$

where $e_j(t) = \sqrt{2/\pi}\sin jt$, $j = 1, 2, \ldots$, $(\cdot,\cdot)$ denotes the scalar product in $L^2(0,\pi)$, P is an orthogonal projection onto $e_m^\perp = \tilde{X}$ (we restrict all these spaces to $C[0,\pi]$). It follows that

$$||(L_0|\tilde{X})^{-1} P|| \le 2 \sum_{j=1, j\neq m}^{\infty} |m^2 - j^2|^{-1}.$$

On the other hand,

$$\gamma = \lim_{|x|\to\infty} \sup_{t\in[0,\pi]} \frac{|f(t,x)|}{|x|}.$$

Put

$$\beta = \frac{2\gamma \sum_{j\neq m} |m^2 - j^2|^{-1}}{1 - 2\gamma \sum_{j\neq m} |m^2 - j^2|^{-1}}.$$

For our purposes, we need $\beta \in (0,1)$, hence

$$4\gamma \sum_{j\neq m}^{\infty} |m^2 - j^2|^{-1} < 1 \tag{8.10}$$

is necessary. The only problem is with condition (C). The following assumption implies it:

(C1) the function f is upper bounded on $\{(t,x) : \sin mt > 0\}$ and lower bounded on $\{(t,x) : \sin mt < 0\}$ and the sets

$$\{t : \sin mt > \beta, \lim_{x\to\pm\infty} f(t,x) = -\infty\} \cup \{t : \sin mt < -\beta, \lim_{x\to\pm\infty} f(t,x) = +\infty\}$$

have positive measures (the first set is obtained if we take limits when $x \to +\infty$, the second – when $x \to -\infty$).

For $\gamma = 0$ (f sublinear), the same condition is sufficient with $\beta = 0$.

Theorem 8.2. *Under the above assumptions, problem 8.9 has no solution for $s \le -s_0$ and has at least two solutions for $s \ge s_0$, where s_0 is some positive constant.*

For example, the following functions are appropriate:

$$f(t,x) = -\gamma x \max(\sin mt - \beta, 0),$$

$$f(t,x) = -\frac{|x|^\delta}{x^2+1} r(t), \quad \delta \in (0,2),$$

where $r : [0,\pi] \to \mathbb{R}$ is a continuous function positive on $(0,\pi/m)$ and vanishing outside this interval.

When $m = 1$ and f is sublinear, we only need

$$\lim_{x\to\pm\infty} \int_0^\pi f(t,x)\sin t\, dt = -\infty$$

which is true for $f(t,x) = g(x)r(t)$ provided that g and r are continuous,

$$\lim_{x\to\pm\infty} g(x) = -\infty, \qquad \lim_{x\to\pm\infty} |g(x)|/|x| = 0,$$

$$\int_0^\pi r(t)\sin t\, dt > 0.$$

If g has a linear growth ($\beta > 0$), then r should vanish on the set $\{t : \sin t \le \beta\}$.

Similar considerations remain true for the problem

$$\Delta u - \lambda_m u = f(x,u) + h_1(x) + t h_0(x), \qquad u|\partial\Omega = 0, \tag{8.11}$$

where λ_m is an eigenvalue with the 1-dimensional eigenspace spanned by h_0. Boundary value problems such as 8.9 and 8.11 were studied by many authors. They extensively used the fact that the eigenvalue is the first one which implies not only the dimension 1 of the eigenspace but also the positiveness of h_0 which was important for their arguments. We omit the last assumption.

The method can be applied to the periodic problem for the second-order scalar equation

$$x'' = f(t, x, x') + h_1(t) + s, \qquad x(0) = x(T), \quad x'(0) = x'(T), \tag{8.12}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function T-periodic with respect to the first variable, h_1 is T-periodic and continuous with

$$\int_0^T h_1(t)\, dt = 0,$$

and s is a real parameter. We can work in the spaces $Z = C_T$ (the space of T-periodic continuous functions), $X = C_T^1$, $Y = C_T^2$, with operators $Lx = x''$, $N(x)(t) = f(t, x(t), x'(t))$. Here, $w(t) = 1$, $\langle u, z\rangle = \int_0^T z(t)\, dt$, $Pz = z - \langle u, z\rangle$ and the Green operator is obtained by the Fourier decomposition

$$(L|\tilde{X})^{-1}Pz(t) = -\frac{T}{2\pi^2}\int_0^T \sum_{k=1}^{\infty} k^{-2}\cos\frac{2k\pi(t-s)}{T} z(s)\, ds.$$

We also need a growth condition for f :

$$|f(t, x, y)| \leq a(|x| + |y|) + b \tag{8.13}$$

that implies $\gamma \leq a$. The upper bound for the norm of the Green operator can be found easily:

$$||(L|\tilde{X})^{-1}P|| \leq \frac{T}{2\pi^2}\sum_{j=1}^{\infty} k^{-2} = \frac{T}{12}.$$

We pass to condition (C). To have it as a property of f, the sign of $\tilde{x}(t) + d$ should depend on the sign of d only. Since $||\tilde{x}|| \leq \beta|d|$, it is necessary to have $\beta < 1$ or, equivalently, $\gamma < \frac{1}{2}$. Hence one should suppose that

$$aT < 6 \tag{8.14}$$

and there exists $\beta < 1 - \frac{aT}{6}$ such that

$$\lim_{|x|\to\infty} \sup_{|y|\leq\beta(1-\beta)^{-1}|x|} \int_0^T f(t, x, y)\, dt = -\infty. \tag{8.15}$$

Theorem 8.3. *The periodic problem 8.12 has no solution for sufficiently large negative s and has at least two solutions for large positive s provided that 8.13, 8.14 and 8.15 hold.*

The same result can be obtained for the periodic problem

$$x'' + cx' = f(t, x, x') + h_1(t) + s, \qquad x(0) = x(T), \quad x'(0) = x'(T)$$

(comp. [21]) with a slight change. One can compute the Green operator

$$(L|\tilde{X})^{-1}Pz = -\sum_{k=1}^{\infty} \frac{T^2}{T^2c^2 + 4k^2\pi^2} \Bigg(((z, e_k) + \frac{Tc}{2k\pi}(z, \hat{e}_k))e_k$$

$$+(-\frac{Tc}{2k\pi}(z, e_k) + (z, \hat{e}_k))\hat{e}_k \Bigg)$$

where $e_k(t) = \sqrt{2/T} \sin k\frac{2\pi}{T}t$, $\hat{e}_k(t) = \sqrt{2/T} \cos k\frac{2\pi}{T}t$ and $(\cdot, \cdot)$ stands for the scalar product in L^2.

There are a lot of functions f satisfying our assumptions. For example, all functions of the form

$$f(t, x, y) = \alpha(t)g(x) + \beta(t, x, y),$$

where α, g, β are continuous, T-periodic with respect to t,

$$\int_0^T \alpha(t)\, dt < 0,$$

$|g(x)| \leq a|x| + b$, $\lim_{|x|\to\infty} g(x) = \infty$ and β is bounded, are appropriate.

The method is applicable for the Neuman problem

$$x'' = f(t, x, x') + h_1(t) + s, \qquad x'(0) = 0 = x'(T),$$

giving very similar results.

Using the arguments from the proof of our main theorem, it is easy to see that the infimum of the set of t's such that equation 8.3 has a solution belongs to this set. This set is, in fact, a closed half-line $[t_1, \infty)$. In most papers, for any $t > t_1$, there exist exactly two solutions. Such a result cannot be obtained under asymptotic assumptions only and by using topological methods. It is geometric in nature.

Analogous arguments when the kernel of the linear part L_0 is n-dimensional lead to the existence of at least 2^n solutions. One can find it in [71].

9 The general perturbation method

We find the much difficult problem when L is not a Fredholm operator (equations with Fredholm linear part of nonzero index were studied by Nirenberg [59] but they are not so important from applicational point of view). This means that either ker L is infinite dimensional, or im L has infinite codimension or im L is not even closed subspace of Z. The third situation seems to be the most difficult but this is the case for many boundary value problems on unbounded domains. It is surprising that the perturbation method developped in the above mentioned paper [63] for Fredholm linear part works here, as well.

The method is applicable if our abstract equation (1.1) can be embedded into a continuos family $L(\lambda)$ and, for $\lambda > 0$ and small (or $\lambda < 0$), the linear operator

$L(\lambda)$ is invertible. Usually, its inverse is not compact because L is non-Fredholm, hence, in order to obtain the solution x_n of $L(\lambda_n)x = N(x)$ for $\lambda_n \to 0$, we need assumptions guaranteeing $L(\lambda_n)^{-1}N$ to be compact and has a fixed point x_n. This is the first group of assumptions. The second one is necessary to get that the sequence (x_n) cannot be unbounded. They rely on conditions of the asymptotic behaviour of $N|\ker L$ and are of Landesman-Lazer type. The third question is if the sequence (x_n) which is bounded, is relatively compact. Usually, it does not need additional assumptions and any cluster point x of this sequence is a solution of 1.1. We shall show three examples of the above procedure.

In the paper [40], W. Karpińska studied the existence of solutions to an ordinary differential equation of the first order which are bounded on the whole line. This question can be considered as the boundary value problem:

$$x' = Ax + f(t,x), \quad x \text{ bounded on } \mathbb{R}, \tag{9.1}$$

where A is a linear selfadjoint operator on $\mathbb{R}^k$ with eigenvalue 0, $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous. The space $\mathbb{R}^k$ can be represented as a direct sum of linear invariant subspaces: X_+ where A has positive eigenvalues, X_- where it has negative eigenvalues, and X_0 – its kernel. Let f_+, f_- and f_0 stand for respective superpositions of f with projectors onto these subspaces.

Theorem 9.1. *If f is bounded,*

(a) $\lim_{t\to\pm\infty} ||f(t,x)|| = 0$ *uniformly on any ball,*

(b) the scalar product $(x, f_0(t,x)) \le 0$ for vectors x with large projections on X_0,

then the problem (9.1) has a solution.

Theorem 9.2. *If f satisfies (a), (b) and (instead of the boundedness)*

(c) $(x, f_+(t,x)) \ge 0$ *for x with large projections on X_+,*

(d) $(x, f_-(t,x)) \le 0$ *for x with large projections on X_-,*

then the problem (9.1) has a solution.

The problem is examined in the space of bounded and continuous functions $x : \mathbb{R} \to \mathbb{R}^k$ denoted by $BC(\mathbb{R}, \mathbb{R}^k)$ with the supremum norm; the linear part $L : x \mapsto x' - Ax$ with the domain $\mathrm{dom}L = \{x \in BC(\mathbb{R}, \mathbb{R}^k : x \in C^1\}$. The role of the Landesman-Lazer type condition plays assumption (b). The existence of solutions to perturbed equations is obtain by using the Schauder Fixed Point Theorem in the case of Theorem 9.1 and the Leray-Schauder degree in the case of Theorem 9.2. The results of [40] are formulated for general Hilbert space instead of $\mathbb{R}^k$ but we restrict ourselves for simplicity here.

Karpińska studied separately [41] the case of second order systems and its bounded solutions. This problem is not a special case of 9.1 – it is the existence of bounded with the first derivative solutions.

In [73] we look for a solution of the nonlinear parabolic system

$$v_t = \Delta v - f(v, a \cdot x - ct) \tag{9.2}$$

where $x, a \in \mathbb{R}^l$, $|a| = 1$, $c > 0$, $v = (v^1, \dots, v^k)$, $\Delta v = (\sum_{j=1}^{l} v^i_{x_j x_j})_{i=1}^k$, $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$. This solution is supposed to be of a special form

$$v(x,t) = w(a \cdot x - ct)$$

with $w : \mathbb{R} \to \mathbb{R}^k$ having finite limits

$$\lim_{s \to \pm\infty} w(s) = \pm w_\pm$$

and is called a travelling wave (of the front wave type). Usually, f depends on v only, and the speed c of the wave and its direction a is not determined by the system (comp. [88]).

If one substitutes $w = u + \psi$ where $\psi(s) = \omega(s)w_- + (1 - \omega(s))w_+$ with ω – a smooth real function that equals 1 for $s \leq -1$ and 0 for $s \geq 1$, then the function u should satisfy the second order ordinary differential system in $\mathbb{R}^k$

$$u'' + cu' = f(u + \psi(s), s) - \psi''(s) - c\psi'(s)$$

and vanish at $\pm\infty$. This is equivalent to an integral Hammerstein equation on the real line

$$u(t) = -\frac{1}{c}\int_{-\infty}^{t} \mathrm{e}^{-c(t-s)} f(u(s) + \psi(s), s)\, ds - \frac{1}{c}\int_{t}^{\infty} f(u(s) + \psi(s), s)\, ds + w_+ - w_-$$

with the additional condition

$$\int_{-\infty}^{\infty} f(u(s) + \psi(s), s)\, ds = c(w_+ - w_-).$$

This system of equations can be considered as a kind of equations (1.4). The perturbation of the above ODE by λu causes that the linear operator begins invertible and the condition on integral over the whole line is omitted. This is similar as in our abstract scheme from section 2.

Theorem 9.3. *([73]). Under the following assumptions on f :*
1) continuity;
2) $|f(x,s)| \leq \alpha(s)|x|^\rho + \beta(s)$ with $\rho < 1$, α and β vanishing at $\pm\infty$,
$\sup_s |\alpha(s)| \leq \alpha_0$ with the constant α_0 sufficiently small;
3) there exists a function γ_0 vanishing at infinity such that for every coordinate f_i of f, $i = 1, \dots, k$, every s, $|u_i| \geq \gamma_0(s)$, and every $|u_j| \leq |u_i|$ $(j \neq i)$

$$u_i f_i(u + \psi(s), s) \geq 0,$$

the parabolic system 9.2 has a solution being of the above form.

In the proof, we perturb the above ODE by $\lambda_n u$ with positive $\lambda_n \to 0$. The related question is reduced to the fixed point problem for some compact operator in the space $C_0(\mathbb{R}, \mathbb{R}^k)$ of functions $u : \mathbb{R} \to \mathbb{R}^k$ vanishing at both infinities. The sequence u_{λ_n} of fixed points is then relatively compact in the above space and any cluster point is a solution of the main problem. Assumption 3) is the Landesman-Lazer type condition and excludes the unboundedness of the sequence.

R. Stańczy [83] considers the question of the existence of bounded solutions for semilinear elliptic problem:

$$\begin{aligned} &\Delta u = f(x,u) \ \text{ for } \ |x| > 1, \ x \in \mathbb{R}^k, \ k \geq 3 \\ &u(x) = 0 \ \text{ for } \ |x| = 1. \end{aligned} \tag{9.3}$$

The problem is resonant, since the homogeneous BVP:

$$\Delta u = 0 \ \text{ for } \ |x| > 1,$$

$$u(x) = 0 \ \text{ for } \ |x| = 1,$$

has a nontrivial bounded solution $u(x) = 1 - |x|^{2-n}$. The Laplace operator is a natural candidate for linear part L, but there is no natural choice of Banach spaces X and Z – the space of bonded and continuous functions is too large and Hölder spaces on unbounded domains are not uniquely defined. If nonlinear part f has the radial symmetry, i.e. $f(x,u) = g(|x|,u)$ where $g : [1,\infty) \times \mathbb{R} \to \mathbb{R}$, then the problem 9.3 leads to

$$v'' + \frac{k-1}{r} v' = g(r,v), \quad v(1) = 0.$$

When v is a solution of the last problem, then $u(x) = v(|x|)$ is a solution of 9.3 called a radial solution. The complete answer to the question of the existence of radial solutions gives the following

Theorem 9.4. *([83]). Suppose that the function g is continuous,*

(i) for each $R > 0$,

$$\lim_{r\to\infty} \sup_{|v|\leq R} |g(r,v)| = 0,$$

(ii) there exists $M > 0$ such that, for all $|v| \geq M$ and all r,

$$vg(r,v) \geq 0.$$

Then the boundary value problem (9.3) has a bounded radial solution.

The proof is based on the perturbation scheme; assumption (ii) plays the role of Landesman-Lazer type condition (notice that it is not asymptotic). When nonlinear term f is not radially symmetric, the question is much more complicated. The perturbed linear operator $\Delta - \lambda I$ is invertible in appropriate Hölder spaces but the boundedness of a sequence of solutions (u_n) for $\lambda_n \to 0$ is not obvious. However, it seems almost sure that the existence of a bounded solution to 9.3 can be obtained under very similar conditions as in Theorem 9.4. The question of the existence of decaying at infinity solutions is simpler (comp. [84] and its references).

Recently, K. Szymańska [85] studied the asymptotic BVP on the half-line at resonance:

$$x'' = f(t,x,x'), \qquad x'(0) = 0 = \lim_{t\to\infty} x'(t).$$

The perturbation of the linear part x'' by $\pm\lambda x$ or $\pm\lambda x'$ does not lead to satisfactory results. One can perturb, however, the boundary condition

$$x'(0) = \lambda x(0), \qquad \lim_{t\to\infty} x'(t) = 0.$$

The perturbed problems are studied in the space $\mathbb{U}$ of functions belonging to $C^1([0,\infty),\mathbb{R}^k)$ which has the limit 0 at infinity with the norm

$$||x|| := \max\{|x(0)|, \sup_{t\in[0,\infty)} |x'(t)|\}.$$

The nonlinearity $f : \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k \to \mathbb{R}^k$ has a linear growth

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + c(t)$$

with $b, c, t \mapsto (t+1)a(t) \in L^1(0,\infty)$. For any $\lambda > 0$, the perturbed problem has a solution provided that

there exists $M > 0$ such that $(y, f(t,x,y)) >$ for $t \ge 0$, $x \in \mathbb{R}^k$ and $|y| \ge M$.

We can apply our method and take x_{λ_n} – a sequence of solutions for $\lambda_n \to 0$ which has a convergent subsequence, if one of two conditions holds:

(1) there exists $L > 0$ such that $(x, f(t,x,y)) \ge \beta t^{\varepsilon}$ for $t \ge 0$, $y \in \mathbb{R}^k$ and $|x| \ge L$;

(2) there exists $L > 0$ such that $(x, f(t,x,y)) \ge 0$ for $t \ge 0$, $y \in \mathbb{R}^k$, $|x| \ge L$, and for some $T > 0$ and $\alpha > 0$, this scalar product is not less than α.

Under the above assumptions, the limit of this subsequence is a solution of the resonant problem.

There is no abstract condition of Landesman-Lazer type for the case of non-Fredholm linear part L similar to the condition given in section 3 on asymptotic behaviour of N restricted to the kernel. However the method works well as we have seen in the above three applications.

If one can find two (or more) separated sequences (x_n), then its cluster points be two disjoint solutions of equation 1.1 (see [70]). Similarly, if one knows that all elements of the sequence (x_n) belongs to a certain closed set, then its cluster points have the same property.

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